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INTERTEMPORAL DECENTRALIZATION*

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The classic papers of Malinvaud and Samuelson pointed out that even in »classical» infinite horizon economies a competitive program of resource allocation need not be efficient or Pareto optimal. This led to the more general question of designing allocation mechanisms in the sense of Hurwicz that are decentralized intertemporally and have appealing normative properties. The paper first reviews some recent results on this topic and goes on to survey properties of evolutionary processes that do not involve communication with future agents.

1. Introduction

The problem of achieving an optimal allocation of resources in a decentralized, infinite horizon economy with finitely-lived agents has been repeatedly raised in the literature. Initially, studies by Malinvaud (1953) and Samuelson (1958) [for a recent assessment, see Malinvaud (1987)] indicated that even in a »classical» infinite horizon economy, a competitive program may be inefficient or Pareto non-optimal. It is only recently that the problem of informational decentralization in infinite economies has been explored formally by using concepts from the literature on designing resource allocation mechanisms. In this review, we provide an exposition of the main results in the framework of an aggregative model. Shorn of technicalities, such a model seems to provide one of the simpler frameworks for isolating those features of a dynamic economy that are critical in limiting the possibilities of decentralization of decision making. Throughout this paper we shall make restrictive assumptions to facilitate statements of results in the simplest and sharpest form.

In Section II we collect some basic concepts from the literature on mechanism theory. The formal definition of a mechanism that is convenient to use in our context is due to Mount and Reiter (1974), and, for us, it is best interpreted in terms of a *verification scenario.* In Section III we recall some basic results from the literature on intertemporal allocation. We provide conditions under which programs that are »optimal» according to some evaluation (social goal) criterion can be shown to exist. In Section IV we examine the relationship between such optimal programs and those that can be »supported» by »competitive» or »shadow» prices by postulating an appropriate version of a »transversality condition».

These earlier results led to a discussion on the possibility of attaining optimal infinite programs in a »decentralized» manner. Indeed, our point of departure is the assessment

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of the contribution of Malinvaud (1953) by Koopmans (1957). Koopmans felt that the typical 'transversality' condition which is sufficient to guarantee efficiency of competitive programs was not 'decentralizable'. In Section IV we review the recent results which indicate that in a stationary environment one can identify optimality of competitive programs through a sequence of period-by-period verifications instead of the asymptotic transversality condition. The additional messages or conditions that can completely characterize optimal competitive programs depend on the structure of the particular model. In Section V we provide alternative period-by-period verification conditions which also characterize the optimality of competitive programs; this characterization is new. In Section VI we discuss intertemporal decentralized mechanisms formally and provide examples of some mechanisms which can realize the social goal of optimality by using the results of Sections III – V. In Section VII we consider nonstationary environments where the producer has incomplete information about the future evolution of technologies, and report an impossibility result on the realization of optimal allocations by any mechanism in this framework.

In a static framework, if an auctioneer begins with an »equilibrium» message, one can visualize the participants responding to such a message »quickly», and a corresponding equilibrium allocation ('action' or quantity decisions) can be implemented. In an infinite horizon economy, not all the agents can be 'assembled' to verify equilibrium conditions in response to a proposed message. Thus, even if we start with an equilibrium message, if one requires the responses of all the agents before allocations can be carried out, no action takes place at all! This difficulty has led to the recent development of a notion of »evolutionary processes» where consumption/investment decisions are carried out period after period on the basis of behavior rules that do not require communication with future [»as yet unborn»] agents. In Section VIII we briefly review this relatively unexplored area. Finally, we have some bibliographical notes on the literature.

It has been observed by Radner (1972, p. 188) that »we may think of decentralization as a special case of division of labor, where »labor» in question is that of making decisions.» The organizer can regard the members of the organization as »machines», receiving messages as inputs and producing messages and actions as outputs. »Beyond this, the organizer can utilize members also to *produce strategies* and even *modification of the organization.*» Viewed this way, »an organization is *information-decentralized* to the extent that different members have different information, and *authority-decentralized* to the extent that individual members are expected (by the organizer) to choose strategies and/or modify the rules of the game.» The scope of the present essay is quite narrow: we do not consider questions related to authority-decentralization, or incentives to follow specific rules of the game. In our model, the agents are given specific rules for verifying certain »myopic» equilibrium conditions. Our primary task is to explore the extent to which a sequence of such short-run verifications based on limited information can achieve some long run goals.

2. *Decentralized Resource Allocation Mechanisms*

Development of a theoretical framework that is broad enough to evaluate and compare alternative economic systems has been a primary motivation behind the research on resource allocation mechanisms. The point of departure for many directions is the Walrasian model of equilibrium and the auctioneer-guided tatonnement process describing a phase of message exchange prior to the attainment of an equilibrium. In a »classical» environment (absence of externalities, indivisibilities and increasing returns etc.) sufficient conditions for the existence of such an equilibrium have been elaborated, and an equilibrium is shown to be Pareto optimal and unbiased. However, to the extent that indivisibilities or increasing returns play an important role in a particular economy, one is left with the question of how best to organize the allocation of resources in such situations to meet specific standards of performance.

The difficulties of achieving an equilibrium through a 'tatonnement' have also been pointed out. The examples of Scarf (1960) and Gale (1963) showed that a tatonnement need not converge at all, or need not converge to a 'fair' equilibrium. Indeed, in a discrete time formulation, one can construct examples of tatonnement displaying 'robust' topological chaos and ergodic chaos in a class of economies even with two goods (Bala and Majumdar (1990)). Furthermore, since no trade can take place out of equilibrium, if convergence does not occur in finite time, no trade is possible in finite time [see Arrow-Hahn (1971) and Fisher (1989) for detailed discussions]. All this has stimulated alternative non-Walrasian equilibrium concepts as well as non-tatonnement processes [see Mukherji (1990) for a recent exposition].

Another prominent feature of the Walrasian model is the 'decentralized' nature of decision making which, it is claimed, 'utilizes incentives' and achieves 'economies of information handling'. While excellent informal expositions of the various related issues have been available since the market socialism debate, a rigorous axiomatic analysis of many aspects has been the principal task and accomplishment of what has been called the »new[2] welfare economics» (Reiter (1986)). We shall now provide some formal definitions (following Mount and Reiter (1974)) related to resource allocation mechanisms. This will be followed by a somewhat informal discussion of the various concepts involved.

We consider the following objects to be given: a *set of agents,* I, a *set of environments,* $E \subset \prod_{i \varepsilon I} E_i$ [where E_i is the set of environments of agent $i \varepsilon I$], and a *space of allocations,* A.

A *mechanism,* π, is a triple (M, ψ, H) where:

(a) M, called the *message space,* is a set of admissible messages.
(b) ψ, called the *equilibrium correspondence,* is a mapping from E to M, such that $\psi(e)$ is non-empty for each $e \varepsilon E$.
(c) H, called the *outcome function,* is a mapping from M to A.

Given a mechanism π, the *performance correspondence,* ϕ, is a mapping from E to A, defined by

$$\phi(e) = \{H(m) : m \varepsilon \psi(e)\}$$

A *social goal correspondence,* Q, is a mapping from E to A, such that Q(e) is non-empty for each e in E.

A mechanism π *realizes* the social goal Q if $\phi(e) \subset Q(e)$ for all $e \varepsilon E$. It is *unbiased* with respect to the social goal Q if $\phi(e) \supset Q(e)$.

A mechanism $\pi = (M, \psi, H)$ is *privacy preserving* if there exist correspondence ψ_i from E_i to M for $i \varepsilon I$, such that

$$\psi(e) = \bigcap_{i \varepsilon I} \psi_i(e_i)$$

for each e in E. (That is, ψ is a »coordinate correspondence»). A mechanism which is privacy preserving will be called a *decentralized mechanism.* We will be concerned only with decentralized mechanisms in what follows.

Given the formal definitions, one can turn to a discussion of how a decentralized mechanism is supposed to operate: this is referred to as the *verification scenario.* Paraphrasing from Hurwicz (1986), the participants in the economy are presented with a proposed message m in M. Agent i accepts the message (says »yes») if and only if m is in $\psi_i(e_i)$. Notice that the response of agent i depends only on the message received, and the information about its own characteristic, e_i, rather than the entire environment, e. This »privacy preserving» property is an essential part of informational decentralization, an aspect we wish to emphasize in this study. A message m is declared an *equilibrium message* if and only if every agent accepts the message; that is, m is in $\psi(e)$.

We now make two brief remarks regarding the »equilibrium message» discussed in the above paragraph. First, if m is not in $\psi_i(e_i)$, agent i rejects the message (says »no»). In this case, a new message must be proposed and the process continued until one is found for which everyone says »yes». How an equilibrium message is found is a topic of considerable importance. However, we will not be directly concerned with it here. [For discussions of message adjustment processes, using »response functions» of agents, yielding an equilibrium message in the limit, see Hurwicz (1986); for an alternative approach, see Reiter (1989)].

Second, it is worth noting that in this interpretation of the operation of the mechanism, there is a limited role of the agent, namely accepting or rejecting a proposed message, given the correspondence ψ_i. The choice of these correspondences as an outcome of the pursuit of self interest (incentive compatibility) is a topic which has been studied in some detail in the literature. Again, we will not be addressing this issue in what follows. [For a discussion, where incentive compatibility means that the response functions ψ_i of the agents have to constitute a Nash equilibrium point of a non-cooperative game, see Hurwicz (1972)].

We continue, now, with our discussion of the operation of the mechanism. If m is an equilibrium message, then the function H specifies an equilibrium allocation, H(m), of this economy, an an outcome of this mechanism. Thus, it is to be understood that this equilibrium allocation is actually carried out. The »performance» of the mechanism can, therefore, be summarized by the set of its equilibrium allocations; this is precisely conveyed by the performance correspondence, ϕ.

The social goal correspondence, Q, specifies a set of allocations judged to be »socially desirable». It is important to note that this correspondence is defined independent of any mechanism under consideration. For each specification of an environment $e \varepsilon E$, Q specifies a non-empty set of allocations $a \varepsilon A$, the attainment of which should be the aim of a constructed mechanism.

The performance of a mechanism is evaluated with respect to the social goal correspondence. If every equilibrium allocation of the mechanisms is a socially desirable allocation, then the mechanism »realizes» the social goal. It is, of course, possible for a mechanism to realize a social goal but be »biased» in the sense that some socially desirable allocations cannot be attained by the mechanism. If every socially desirable allocation is an equilibrium allocation of the mechanism, the mechanism is »unbiased» with respect to the social goal.

3. *An Aggregative Model of Intertemporal Allocation*

3.1 Notation

The set of non-negative integers is denoted by $N = \{0, 1, 2, \ldots\}$; $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$) denotes the set of non-negative reals (resp. positive reals). A sequence $x = (x_t)$ of reals is *non-negative* (written $x \geq 0$) if $x_t \varepsilon \mathbb{R}_+$ for all t in N; x is *strictly positive* (written $x \gg 0$) if $x_t \varepsilon \mathbb{R}_{++}$ for all t in N; x is *positive* (written $x > 0$) if x is non-negative and $x_t \varepsilon \mathbb{R}_{++}$ for some t. For any two infinite sequences $x = (x_t)$ and $x' = (x'_t)$, $x \geq$ (resp. $>$, $\gg$), x' if $x - x' \geq$ (resp. $>$, $\gg$) 0. The set of all non-negative (resp. positive, strictly positive) sequences is denoted by S_+ (resp. $\bar{S}$, S_{++}). The T-fold Cartesian product of A with itself is written A^T (when T is infinite, we write $A^{(\infty)}$). Given any infinite sequence x, $x^{(T)}$ denotes the (T+1) vector whose elements are the first (T+1) elements of x.

3.2 A Stationary Economy

Consider an infinite horizon, one-good model with a stationary *technology* described by a gross output function f which satisfies the following regularity conditions [F]:

[F.1] $f: \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous on* $\mathbb{R}_+$;
[F.2] $f(0) = 0$.
[F.3] $f(x)$ *is twice differentiable at* $x > 0$, *with* $f'(x) > 0$ *and* $f''(x) < 0$.
[F.4] (a) $\lim_{x \downarrow 0} f'(x) = \infty$; $\lim_{x \uparrow \infty} f'(x) = 0$
(b) $f(x) \to \infty$ as $x \to \infty$

Note that the assumptions imply that f is strictly increasing and strictly concave on $\mathbb{R}_+$.

Given the initial stock $y > 0$, a program of resource allocation [briefly, »a program (x, y, c)»] from y consists of non-negative sequences of *inputs* $x = (x_t)$, *stocks* $y = (y_t)$ and *consumptions* $c = (c_t)$ satisfying

$$(3.1) \quad y_0 = y, \; y_{t+1} = f(x_t) \text{ and } y_t = x_t + c_t \quad \text{for } t \geq 0$$

Observe that the introduction of a production lag implies that the decision in period t on how much to consume (choice of c_t) and how much to use as input (choice of x_t) limits the choice possibilities of all periods subsequent to period t, whereas such decisions have no bearing on choices made in the preceding periods.

Corresponding to any f satisfying (F.1) – (F.4) there exists a unique number $K > 0$, such that

(3.2) (i) $f(K)=K$;
(ii) $f(x)>x$ for $0<x<K$;
(iii) $f(x)<x$ for $x>K$

Indeed for any program (x, y, c) from $y>0$ one has

(3.3) $y_t \leqq K(y) \equiv \max(K, y)$ for $t \geq 0$

We refer to K as the *maximum sustainable stock* given f.

Given any δ satisfying $0<\delta<1$, there is a unique positive solution x^*_δ to the equation

$$\delta f'(x)=1$$

Write $y^*_\delta = f(x^*_\delta)$, $c^*_\delta = y^*_\delta - x^*_\delta$. When $\delta = 1$ (resp. $0<\delta<1$), we refer to $(x^*_\delta, y^*_\delta, c^*_\delta)$ as the *golden rule* (resp. *modified golden rule*) input, stock and consumption. Given $0<\delta\leq 1$, one can define the stationary program (x^*, y^*, c^*) by

(GR) $x^*_t = x^*_\delta$, $y^*_t = y^*_\delta$, $c^*_t = c^*_\delta$ for $t \in N$

When $\delta=1$ (resp. $0<\delta<1$), this stationary program (GR) is referred to as the *golden* (resp. *modified golden*) *rule program*.

Alternative feasible programs are evaluated according to some criterion which reflects »social goals». We shall review results on realizing some well-known criteria.

3.3 Efficiency

A program (x, y, c) from $y>0$ is *efficient* if there is no program (x', y', c') from the same initial stock y such that $c'_t \geq c_t$ for all $t \geq 0$ and $c'_t > c_t$ for some t. It is easy to see that there are infinitely many efficient programs from any $y>0$.

Characterization of efficient programs in terms of shadow prices was the central theme of Malinvaud (1953, 1961). Let us define an *intertemporal profit maximizing program* from y as a sequence (x, y, c, p) such that (x, y, c) is a program from y, $p=(p_t)$ is a positive sequence, and for all $t \in N$,

(M) $p_{t+1} f(x_t) - p_t x_t \geqq p_{t+1} f(x) - p_t x$ for $x \geqq 0$

Here, p_t is interpreted as the *discounted price* of the good in period t. The prices $p=(p_t)$ are said to 'support' the program (x, y, c) and will be called *Malinvaud prices*. Since the sequence p is assumed positive, p_0 is positive (using (M) and the assumption that f is strictly increasing on $\mathbb{R}_+$). If the input sequence $x=(x_t)$ is strictly positive so is the price sequence $p=(p_t)$ and one derives the familiar 'marginal' condition:

(3.4) $p_{t+1} f'(x_t) = p_t$

Example 3.1

Consider the sequence $(\hat{x}, \hat{y}, \hat{c})$ from $y>0$ defined by:

$$y=\hat{y}_0=\hat{x}_0,\ \hat{y}_t=\hat{x}_t=f(\hat{x}_{t-1}) \text{ for } t\geq 1,\ \hat{c}_t=0 \text{ for } t\geq 0$$

This is clearly a program: it is called the *program of pure accumulation* because it prescribes zero consumption in all periods. Since $y>0$, this program is not efficient. However, the input sequence $\hat{x}$ being strictly positive, one can set $\hat{p}_0=1$, and define $\hat{p}_{t+1} = f'(\hat{x}_t)\,\hat{p}_t$ for all $t\geq 0$. This sequence p gives us Malinvaud prices relative to which the program of pure accumulation satisfies the condition (M).

In fact, one can use the above reasoning to establish a fairly general result in our framework.

Proposition 3.1:

If (x, y, c) *is a program from* $y>0$, *then there exists a sequence* $p=(p_t)$ *of discounted prices such that* (x, y, c, p) *is an intertemporal profit maximizing program from* y.

Proof:

To see this, note that if $x=(x_t)$ is a strictly positive sequence, then we can define a sequence $p=(p_t)$ as follows:

(3.5) $p_0=1$, $p_{t+1}=[p_t/f'(x_t)]$ for $t \in N$

It follows from concavity of f that given any $x\geq 0$ and $t \in N$, $f(x)-f(x_t)\leq f'(x_t)(x-x_t)$, so that $p_{t+1}[f(x)-f(x_t)]\leq p_t(x-x_t)$, using (3.5). This yields condition (M) after transposing terms.

If $x=(x_t)$ is not a strictly positive sequence, let $T \geq 0$ be the first period for which $x_T = 0$. Then $x_t = 0$ for $t \geq T$ by (F.2). In this case, one can define $p = (p_t)$ as follows:

$$\left.\begin{array}{lll} p_0 = 1,\ p_t = 0 & \text{for } t \geq T+1 & \\ p_{t+1} = p_t/f'(x_t) & \text{for } 0 \leq t \leq T-1 & \text{if } T \geq 1 \end{array}\right\}$$

It is easy to check, as above, that (x, y, c, p) is an intertemporal profit maximizing program.//

Remark:

Consider a production function, f, which satisfies (F.1), (F.2) and is nondecreasing and concave on $\mathbb{R}_+$. Then, we have what might be called a »classical» environment, since the technology set $\{(x, y) \in \mathbb{R}^2_+ : y \leq f(x)\}$ would be closed, convex and allow free-disposal. In this more general framework, Proposition 3.1 is not valid. Indeed, even if we restrict the program (x, y, c) in question to be *efficient,* the result fails to hold. This was demonstrated convincingly by McFadden (1975), who constructed an example of an efficient program, whose only supporting prices (in the sense of (M)) is the null sequence.

Example 3.2 (McFadden)

Suppose that the production function is:

$$\text{(3.6)}\quad f(x) = \begin{cases} 2 - 2(1-x)^2 & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } x \geq 1 \end{cases}$$

Let J be the set of time periods $t_n = 3^n$, $n = 1, 2, 3 \ldots$. Define the sequence (x, y, c) as follows:

$$\text{(3.7)}\quad x_t = 1 \text{ if } t \in J,\ x_t = (1/2) \text{ if } t \notin J$$

$$y_t = 2 \text{ if } t-1 \in J,\ y_t = (3/2) \text{ if } t-1 \notin J$$

$$c_t = (1/2) \text{ if } t \in J,\ c_t = 1 \text{ if } t-1 \notin J,\ c_t = (3/2) \text{ if } t-1 \in J$$

One can show (see McFadden (1975)) that (x, y, c) is an efficient program from $y = (3/2)$. Since x is strictly positive and $f'(x_t) = 0$ infinitely often, the relationship $p_{t+1} f'(x_t) = p_t$ characterizing intertemporal profit maximization can hold only with $p = 0$.

In McFadden's example, f satisfies [F.1], [F.2]. Furthermore, f is concave and continuously differentiable. However, $f'(x) = 0$ at $x \geq 1$, so it violates [F.3]. In fact what is crucial for the validity of Proposition 3.1 (other than continuity and concavity of f) is the fact that f is *strictly increasing* (our proof is made easier by assuming in addition that f is differentiable, but this is not essential for the result). In a multisectoral model, this condition translates to what Malinvaud (1953) has called the »non-tightness» condition.

Examples 3.1 and 3.2 indicate two startling possibilities in *infinite horizon* economies: even in »classical» environments intertemporal profit maximization (M) in every period does not guarantee efficiency; and an efficient program need not have a system of Malinvaud prices at which such a profit maximization condition (M) can be shown to hold.

Thus the economic implications of the »duality» results on productive efficiency and supporting prices [summarized, for example, by Koopmans (1957, pp. 83–92)] need reappraisal as we make a transition from the finite to the infinite horizon. Perhaps the most important result in this direction is the following one, due to Malinvaud (1953).

Proposition 3.2 (Malinvaud)

Suppose (x, y, c, p) *is an intertemporal profit maximizing program from* $y > 0$, *satisfying* $p_t > 0$ *for* $t \geq 0$, *and*

$$\text{(IF)}\quad \lim_{t \uparrow \infty} p_t x_t = 0$$

Then (x, y, c) *is an efficient program from y.*

Proof:

To see this, let $(\bar{x}, \bar{y}, \bar{c})$ be any program from y. Now for any finite T,

$$\text{(3.8)}\quad \sum_{t=0}^{T} p_t(\bar{c}_t - c_t) = \sum_{t=0}^{T} p_t[(\bar{y}_t - \bar{x}_t) - (y_t - x_t)]$$

$$= \sum_{t=0}^{T-1} [(p_{t+1} f(\bar{x}_t) - p_t\bar{x}_t) - (p_{t+1} f(x_t) - p_t x_t)] - p_T\bar{x}_T + p_T x_T$$

Using the condition (M) for $t = 0, \ldots, T-1$ and non-negativity of $p_T\bar{x}_T$,

$$\text{(3.9)}\quad \sum_{t=0}^{T} p_t(\bar{c}_t - c_t) \leqq p_T x_T$$

If (x, y, c) is *not* efficient, there is a program (x', y', c') from y such that $c'_t \geq c_t$ for all $t > 0$, and $c'_\tau = c_\tau + m$, $m > 0$ for some period τ. Hence, for all $T \geq \tau$

$$(3.10)\quad 0 < mp_\tau \leqq \sum_{t=0}^{T} p_t(c'_t - c_t)$$

By combining (3.9), (3.10) and condition (IF), we get a contradiction. This proves that (x, y, c) is efficient.//

The important condition in Proposition 3.2 is that the present value of input becomes insignificant over time. It is commonly referred to as the *insignificant future* or the *transversality* condition.

In his comments on Propositions 3.1 and 3.2, Koopmans observed the following: »By giving sufficiently free rein to out imagination we can still visualize» the condition (M) of intertemporal profit maximization »as being satisfied through a decentralization of decisions among an infinite number of producers, each in charge of that part of the program relating to one future period. A further decentralization among many contemporaneous producers within each period can also be visualized. But even at this level of abstraction, it is difficult to see how the task of meeting condition» (IF) can be pinned down on any particular decision maker. »This is a new condition, to which there is no counterpart in the finite models.» It should be stressed that if an agent in a particular period is allowed to observe only a finite number of prices and quantities, it is never able to check whether or not condition (IF) is satisfied; of course, even with the observation of an infinite subsequence of $(p_t x_t)$, it may not be possible to verify (IF).

At this juncture we would like to comment upon an alternative approach to study the relationship between efficiency (or Pareto optimality) and value maximization with an infinite dimensional commodity space. With a single good in each period t, one can formally identify the *commodity space* (see Debreu (1959)) with the linear space S of all real sequences. In fact, in view of (3.3), one can restrict one's attention to l_∞, the linear space of all bounded real sequences. For any element $x = (x_t)$ in l_∞, define the norm $||x||$ as:

$$(3.11)\quad ||x|| = \sup_t |x_t|$$

The set of all linear functionals continuous with respect to the »sup norm» topology is denoted by l^*_∞. An element p in l^*_∞ is *non-negative* if $p(x) \geq 0$ for all $x \geq 0$.

Given a program (x, y, c) from some $y > 0$, let us call $c = (c_t)$ a consumption program from y. Denote the set of all consumption programs from $y > 0$ by C. Then C is a nonempty convex set of l_∞. A separation argument can be invoked (see Radner (1967)) to prove the following result.

Proposition 3.3

Let $(\bar{x}, \bar{y}, \bar{c})$ *be an efficient program from* $y > 0$. *Then there is a non-zero, non-negative* p *in* l^*_∞ *such that*

$$(3.12)\quad p(\bar{c}) \geqq p(c)$$

for all c *in* C.

While the value maximization property (3.12) is undoubtedly of some interest, one obvious difficulty is that the continuous linear functional p need not be representable by a sequence $p = (p_t)$ with respect to which the period-by-period rule of profit maximization can be formulated. Indeed, it is known that if p is in l^*_∞ then for any c in l_∞,

$$(3.13)\quad p(c) = \sum_{t=0}^{\infty} p_t c_t + p_\infty(c)$$

where (p_t) is a summable sequence ($\sum_{t=0}^{\infty} |p_t|$ is finite) and $p_\infty(c)$ is obtained by integrating c with respect to a purely finitely additive measure on N (see Radner (1967), Majumdar (1970)). Even when p is non-negative, it is possible that $p_t = 0$ for all $t \in N$. Moreover, $p_\infty(c) = p_\infty(c')$ whenever c and c' differ only over a finite number of time periods. Thus, the valuation rules (3.13) cannot be easily interpreted or implemented in the context of designing an intertemporally decentralized resource allocation mechanism (see Malinvaud (1961)).

Example 3.3

Consider the golden rule program (GR) defined above. With $\delta = 1$, we simply drop the subscript δ and write (x^*, y^*, c^*). One can verify directly that it is an efficient program. Moreover, at the stationary price sequence $p^* = (p^*_t)$ where $p^*_t = 1$ for $t \in N$, it satisfies the condition (M) of intertemporal profit maximization. Observe that $p_t^* x_t^* = x^* > 0$. Hence, the condition (IF) is not satisfied. Clearly p^* is not a summable sequence. Also, the non-negative, non-zero linear functional p at which c^* maximizes value (see (3.12)) has no associated sequence (p_t) in l_1. In fact,

$$p(c^*) = p_\infty(c^*) = c^*$$

Also, for *any* feasible consumption program $c = (c_t)$ such that $\lim_{t \uparrow \infty} c_t$ exists,

$$p(c) = \lim_{t \uparrow \infty} c_t$$

In particular, $p(c) = c^*$ for all feasible consumption programs $c = (c_t)$ such that $\lim_{t \uparrow \infty} c_t = c^*$.

3.4 Optimality

In discussing the evaluation criterion known as »optimality», one typically introduces a *utility* function, u, from $\mathbb{R}_+$ to $\mathbb{R}$, and a *discount factor,* δ, satisfying $0 < \delta \leq 1$. A program $(\bar{x}, \bar{y}, \bar{c})$ from $y > 0$ is *optimal* if

$$(3.14)\quad \limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t \, [u(c_t) - u(\bar{c}_t)] \leq 0$$

for every program (x, y, c) from y. When $\delta < 1$, so that future utilities are given smaller weights than current ones, the optimality exercise is referred to as the »discounted» case; when $\delta = 1$, we refer to the exercise as the »undiscounted» or »Ramsey» case. In what follows we examine the problem of existence of optimal programs in each case, when the utility function, u, satisfies:

(U.1) u *is continuous on* $\mathbb{R}_+$

(U.2) u *is twice differentiable at* $c > 0$ *with* $u'(c) > 0$ *and* $u''(c) \leq 0$.

Clearly, (U.1), (U.2) imply that u is concave and strictly increasing on $\mathbb{R}_+$. In what follows, we normalize $u(0) = 0$.

3.4.1 The Discounted Case

When $0 < \delta < 1$, boundedness of the set of programs from any $y \geq 0$ [recall (3.3)], and continuity of the utility and production functions [(U.1) and (F.1)] can be exploited to obtain the existence of an optimal program.

Proposition 3.3:

There is a unique optimal program from any $y \geq 0$.

Proof:

If (x, y, c) is any program from y, then $0 \leq c_t \leq K(y) \equiv \max(K, y)$ for all $t \geq 0$, and so $0 = u(0) \leq u(c_t) \leq u(K(y))$ for $t \geq 0$. Denote $u(K(y))$ by b. Then, clearly, $0 \leq u(c_t) \leq b$ for $t \geq 0$, and so for every $T \geq 1$

$$\sum_{t=0}^{T} \delta^t u(c_t) \leq b/(1-\delta)$$

Since $u(c_t) \geq 0$ for $t \geq 0$, we know that

$$\sum_{t=0}^{\infty} \delta^t u(c_t) \equiv \lim_{T \to \infty} \sum_{t=0}^{T} \delta^t u(c_t)$$

exists and cannot exceed $[b/(1-\delta)]$. Let $a = \sup\{\sum_{t=0}^{\infty} \delta^t u(c_t)$: (x, y, c) is a program from y}.

Choose a sequence of programs (x^n, y^n, c^n) from y such that

$$(3.15)\quad \sum_{t=0}^{\infty} \delta^t u(c_t^n) \geq a - (1/n) \qquad \text{for } n = 1, 2, 3, \ldots$$

Using (3.3), continuity of f, and the Cantor diagonal method, there is a program $(\bar{x}, \bar{y}, \bar{c})$ from y, and a subsequence of n (retain notation) such that for $t \geq 0$,

$$(3.16)\quad (x_t^n, y_t^n, c_t^n) \to (\bar{x}_t, \bar{y}_t, \bar{c}_t) \quad \text{as } n \to \infty$$

We claim that $(\bar{x}, \bar{y}, \bar{c})$ is an optimal program from y. If the claim is false, then we can find $\varepsilon > 0$ such that

$$\sum_{t=0}^{\infty} \delta^t u(\bar{c}_t) < a - \varepsilon$$

Pick T such that $\delta^{T+1}[(b/1-\delta)] < (\varepsilon/3)$. Using (3.16), and the continuity of u, we can find $\bar{n}$, such that for all $n \geq \bar{n}$,

$$\sum_{t=0}^{T} \delta^t u(c_t^n) < \sum_{t=0}^{T} \delta^t u(\bar{c}_t) + (\varepsilon/3)$$

Thus, for all $n \geq \bar{n}$, we get

$$\sum_{t=0}^{T} \delta^t u(c_t^n) \leq \sum_{t=0}^{T} \delta^t u(c_t^n) + \delta^{T+1}[(b/1-\delta)]$$
$$< \sum_{t=0}^{T} \delta^t u(c_t^n) + (\varepsilon/3)$$
$$< \sum_{t=0}^{T} \delta^t u(\bar{c}_t) + (2\varepsilon/3)$$
$$< a - (\varepsilon/3)$$

But this leads to a contradiction to (3.15) for $n > \max\,[\bar{n}, (3/\varepsilon)]$. This establishes our claim. Uniqueness of the optimal program follows from the strict concavity of f.//

3.4.2 The Undiscounted Case

When $\delta = 1$, the method of proving the existence of an optimal program is a more involved one; the pay-off, however, is that several interesting results are also established simultaneously.

The focus of attention here is the golden-rule program (GR) defined above; to simplify notation we drop the subscript δ. If we define $p^* \equiv u'(c^*)$, we can easily check [using the concacity of f and u] that

$$\text{(3.17)} \quad \left.\begin{array}{ll} u(c^*) - p^*c^* \geq u(c) - p^*c & \text{for } c \geq 0 \\ p^*f(x^*) - p^*x^* \geq p^*f(x) - p^*x & \text{for } x \geq 0 \end{array}\right\}$$

At any $c \geq 0$, define the *consumption value loss at* p^* as:

$$\alpha(c) \equiv [u(c^*) - p^*c^*] - [u(c) - p^*c]$$

Similarly, at any $x \geq 0$, define the *loss of intertemporal profit at* p^* as:

$$\beta(x) \equiv [p^*f(x^*) - p^*x^*] - [p^*f(x) - p^*x].$$

Using (3.17), $\alpha(c) \geq 0$ for all $c \geq 0$ and $\beta(x) \geq 0$ for all $x \geq 0$. Of particular interest for the sequel is the following »value loss lemma»:

Lemma 3.1

Let $\hat{K}$ *be a given positive number. For any* $\varepsilon > 0$, *there is* $\beta > 0$, *such that if* $0 \leq x \leq \hat{K}$ *and* $|x - x^*| \geq \varepsilon$, *then* $\beta(x) \geq \beta$.

Proof:

If the claim is false, there is $\varepsilon > 0$ and a sequence x^n such that $0 \leq x^n \leq \hat{K}$ and $|x^n - x^*| \geq \varepsilon$, but $\beta(x^n) \to 0$ as $n \to \infty$. Consider a subsequence of x^n [retain notation] converging to some $\bar{x}$. Then $0 \leq \bar{x} \leq \hat{K}$ and $|\bar{x} - x^*| \geq \varepsilon$. Also, since $\beta(x^n) \to 0$ and $x^n \to \bar{x}$, continuity of f implies that $\beta(\bar{x}) = 0$. But, then, by strict concavity of f, for $x = (1/2)\bar{x} + (1/2)x^*$, we would have $\beta(x) < 0$, a contradiction.//

Estimates of the *sum* of all value losses on a program can be obtained by using the following result, the proof of which is obvious and therefore omitted.

Lemma 3.2

If (x, y, c) *is a program from* $y > 0$, *then for each finite* $T \geq 1$,

$$\text{(3.18)} \quad \sum_{t=0}^{T} [u(c_t) - u(c^*)] = p^*(y - y^*) - p^*(x_T - x^*) - \sum_{t=0}^{T} \alpha(c_t) - \sum_{t=0}^{T} \beta(x_t)$$

A program (x, y, c) from $y > 0$ is *good* if there exists a real number B, such that

$$\sum_{t=0}^{T} [u(c_t) - u(c^*)] \geq B \quad \text{for all } T \geq 0$$

and it is *bad* if

$$\sum_{t=0}^{T} [u(c_t) - u(c^*)] \to -\infty \quad \text{as } T \to \infty$$

It can be shown, following Gale (1967), that given any $y > 0$, the set of good and bad programs from y exhaust the set of *all* programs from y. Furthermore, the set of good programs is non-empty.

Lemma 3.3

There exists a good program from every $y > 0$. *If a program from* $y > 0$ *is not good, it is bad.*

Proof:

Consider the pure accumulation program $(\hat{x}, \hat{y}, \hat{c})$ from y (recall Example 3.1). It is not difficult to show that $\hat{x}_t \to K$ as $t \to \infty$, where K is defined in (3.2). Since $K > x^*$, $\hat{x}_t > x^*$ for all t sufficiently large. Let τ be the first period such that $\hat{x}_\tau \geq x^*$. Consider the sequence (x, y, c) defined by: $y_0 = y$; $x_t = \min\,(\hat{x}_t, x^*)$, $y_{t+1} = f(x_t)$ and $c_t = y_t - x_t$ for $t \geq 0$. Then (x, y, c) is clearly a program from y, and $c_t = c^*$ for all $t > \tau$. Hence (x, y, c) is a good program.

Suppose, next, that (x', y', c') is a program from $y>0$ which is not good. Then, given any real number B, there is some T, such that

$$\sum_{t=0}^{T} [u(c_t') - u(c^*)] < B$$

Using the fact that $y_t' \leq K(y) \equiv \max(K, y)$ for $t \geq 0$ [recall (3.3)], and Lemma 3.2, we also have for all $\tau > T$

$$\sum_{t=T+1}^{\tau} [u(c_t') - u(c^*)] \leq p^*[K(y) + x^*]$$

Thus, given any real number B, there is some T, such that for all $\tau > T$,

$$\sum_{t=0}^{\tau} [(u(c_t') - u(c^*)] < p^*[K(y) + x^*] + B$$

which shows that (x', y', c') is bad.//

In view of the previous result, our search for an optimal program can be confined to the class of good programs. Good programs converge to the golden-rule in input, stock and consumption levels, a result usually referred to as the »turnpike» property.

Lemma 3.4

Let (x, y, c) *be any good program from* $y>0$. *Then* $(x_t, y_t, c_t) \to (x^*, y^*, c^*)$ *as* $t \to \infty$.

Proof:

Suppose x_t does not converge to x^*. Then there is some $\varepsilon > 0$, and a subsequence of periods for which $|x_t - x^*| \geq \varepsilon$. Using Lemma 3.1 and defining $\hat{K} \equiv K(y)$, there is $\beta > 0$ such that $\beta(x_t) \geq \beta$ for this subsequence of periods. Using Lemma 3.2, it then follows that (x, y, c) is not good. Thus, $x_t \to x^*$ as $t \to \infty$; consequently, $y_t = f(x_{t-1}) \to f(x^*) = y^*$ as $t \to \infty$, and $c_t = y_t - x_t \to (y^* - x^*) = c^*$ as $t \to \infty$.//

Lemma 3.5

If (x, y, c) *is any good program from* $y>0$,

$$\lim_{T \uparrow \infty} \sum_{t=0}^{T} [u(c_t) - u(c^*)] \quad \textit{exists}$$

Proof:

Since (x, y, c) is good, there is a real number B such that for all $T \geq 0$

$$(3.19) \quad \sum_{t=0}^{T} [u(c_t) - u(c^*)] \geq B$$

Using Lemma 3.2 and (3.19), we have for $T \geq 0$,

$$(3.20) \quad \sum_{t=0}^{T} \alpha(c_t) + \sum_{t=0}^{T} \beta(x_t) \leq p^*y + p^*x^* - B$$

Since $\alpha(c_t) \geq 0$ and $\beta(x_t) \geq 0$ for all $t \geq 0$, (3.20) implies that

$$(3.21) \quad L(x, y, c) \equiv \lim_{T \to \infty} \left[\sum_{t=0}^{T} \alpha(c_t) + \sum_{t=0}^{T} \beta(x_t)\right]$$

exists. Also, Lemma 3.4 implies that $p^*x_t \to p^*x^*$ as $T \to \infty$. Using this and (3.21) in Lemma 3.2, one can take the limit in (3.18) to obtain

$$(3.22) \quad \sum_{t=0}^{\infty} [u(c_t) - u(c^*)] = p^*y + p^*x^* - L(x, y, c)$$

This establishes the lemma.//

We can now finally prove the result on the existence of an optimal program.

Proposition 3.4:

There is a unique optimal program from any $y>0$.

Proof:

Let $L(y) = \inf\{L(x, y, c)$: (x, y, c) is a good program from y}. Now, take a sequence (x^n, y^n, c^n) of good programs from y such that

$$(3.23) \quad L(x^n, y^n, c^n) \leq L(y) + (1/n) \quad \text{for } n = 1, 2, 3, \ldots$$

Using (3.3), continuity of f, and the Cantor diagonal method, there is a program $(\bar{x}, \bar{y}, \bar{c})$ from y, and a subsequence of n (retain notation) such that for $t \geq 0$,

$$(3.24) \quad (x_t^n, y_t^n, c_t^n) \to (\bar{x}_t, \bar{y}_t, \bar{c}_t) \quad \text{as } n \to \infty$$

We claim that $(\bar{x}, \bar{y}, \bar{c})$ is good and $L(\bar{x}, \bar{y}, \bar{c}) = L(y)$. If the claim is false, then we can find a positive integer T, and $\varepsilon > 0$, such that

$$(3.25)\quad [\sum_{t=0}^{T} \alpha(\bar{c}_t) + \sum_{t=0}^{T} \beta(\bar{x}_t)] \geq L(y) + \varepsilon$$

Using (3.24) and (3.25), and the continuity of f and u, we can find a positive integer $\bar{n}$, such that for all $n \geq \bar{n}$,

$$(3.26)\quad [\sum_{t=0}^{T} \alpha(c_t^n) + \sum_{t=0}^{T} \beta(x_t^n)] \geq L(y) + (\varepsilon/2)$$

Clearly (3.26) implies that for all $n \geq \bar{n}$, $L(x^n, y^n, c^n) \geq L(y) + (\varepsilon/2)$, which contradicts (3.23) for $n > \max[\bar{n}, (2/\varepsilon)]$. This establishes our claim.

Using Lemma 3.3 and (3.22), it follows readily that $(\bar{x}, \bar{y}, \bar{c})$ is an optimal program from y. The uniqueness of the optimal program follows from the strict concavity of f.//

3.4.3 Value and Policy Functions

Given Propositions 2.2 and 3.4, we can present an elementary treatment of the concepts of »value» and »policy» functions, which are prominent in the dynamic programming approach to intertemporal optimality.

When $0 < \delta < 1$, one can define a value function, $V: \mathbb{R}_+ \to \mathbb{R}$ by

$$(3.27)\quad V(y) = \sum_{t=0}^{\infty} \delta^t u(\bar{c}_t)$$

where $(\bar{x}, \bar{y}, \bar{c})$ is the optimal program from $y \geq 0$. When $\delta = 1$, one can similarly define a *value function*, $V: \mathbb{R}_{++} \to \mathbb{R}$ by

$$(3.28)\quad V(y) = \sum_{t=0}^{\infty} [u(\bar{c}_t) - u(c^*)]$$

where $(\bar{x}, \bar{y}, \bar{c})$ is the optimal program from y.

Since there is a unique optimal program (in both the discounted and undiscounted cases) from every $y > 0$, one can define an *(optimal consumption) policy function,* $g: \mathbb{R}_{++} \to \mathbb{R}_+$ by

$$(3.29)\quad g(y) = \bar{c}_0$$

where $(\bar{x}, \bar{y}, \bar{c})$ is the optimal program from y. It is easily checked that this definition also implies that for all $t \geq 0$, $\bar{c}_t = g(\bar{y}_t)$.

In what follows, we state some of the basic properties of the value and policy functions.

Lemma 3.6:

The value function, V, *is increasing, strictly concave and continuous on* $\mathbb{R}_{++}$; *the policy function,* g, *is continuous on* $\mathbb{R}_{++}$.

While the strict concavity and continuity of the value function follow directly from the strict concavity of f, the continuity of the policy function requires an application of a maximum theorem.

Some additional properties of V and g can be explored in two important subcases of the class of utility function allowed by (U.1) and (U.2): (i) the strictly concave case; (ii) the linear case.

Strictly Concave Utility Function

If we retain (U.1), strengthen (U.2) to:

(U.2$^+$) u *is twice differentiable at* $c > 0$, *with* $u'(c) > 0$, $u''(c) < 0$ and assume, in addition:

(U.3) $u'(c) \to \infty$ *as* $c \to 0$

then we refer to the sub-case, for convenience, as the »strictly concave case». In this sub-case the following basic relationship between the value and policy functions can be established.

Lemma 3.7:

In the strictly concave case (i) *The value function,* V, *is differentiable on* $\mathbb{R}_{++}$; (ii) *the policy function,* g, *satisfies* $0 < g(y) < y$ *for all* $y > 0$; *and* (iii) $V'(y) = u'(g(y))$ *for all* $y > 0$.

Linear Utility Function

If $u(c) = c$ for all $c \geq 0$ (which, of course, satisfies (U.1) and (U.2) we refer to the sub-case as the »linear case». In this sub-case, it is, in fact, possible to describe the policy function, g, explicitly.

Lemma 3.8:

In the linear case, given $0 < \delta \leq 1$, *the policy function,* g *satisfies*

$$g(y) = \max(y - x_\delta^*, 0) \text{ for all } y > 0$$

where x_δ^* *is the golden-rule* ($\delta = 1$) *or modified golden-rule* ($0 < \delta < 1$) *input.*

Proof:

Given $y > 0$ and $0 < \delta \leq 1$, define the sequence $(\bar{x}, \bar{y}, \bar{c})$ by $\bar{y}_0 = y$, $\bar{y}_{t+1} = \min(f(\bar{y}_t), f(x_\delta^*))$ for $t \geq 0$, $\bar{c}_t = g(\bar{y}_t)$ for $t \geq 0$ and $\bar{x}_t = \bar{y}_t - \bar{c}_t$ for $t \geq 0$. It can be checked that this is a program from y. We have to show that this is an optimal program from y.

Let (x, y, c) be any program from y. We consider two cases: (i) $y \geq x^*_\delta$, (ii) $y < x^*_\delta$.

Case (i). We have for any T,

$$\sum_{t=0}^{T} \delta^t (c_t - \bar{c}_t) = \sum_{t=0}^{T-1} \delta^t \{[\delta f(x_t) - x_t] - [\delta f(x^*_\delta) - x^*_\delta]\} + \delta^T (x^*_\delta - x_T)$$

since $y_0 = \bar{y}_0 = y$, and $\bar{x}_t = x^*_\delta$ for $t \geq 0$. Now, for all $x \geq 0$, $\delta f(x) - x \leq \delta f(x^*_\delta) - x^*_\delta$, using the concavity of f and $\delta f'(x^*_\delta) = 1$. Thus, for any T,

$$\text{(3.30)} \quad \sum_{t=0}^{T} \delta^t (c_t - \bar{c}_t) \leq \delta^T (x^*_\delta - x_T)$$

If $0 < \delta < 1$, the right hand side expression in (3.30) converges to zero as $T \to \infty$, establishing the optimality of $(\bar{x}, \bar{y}, \bar{c})$. If $\delta = 1$, and (x, y, c) is good, $x_T \to x^*$ as $T \to \infty$, so that

$$\text{(3.31)} \quad \limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t (c_t - \bar{c}_t) \leq 0$$

If $\delta = 1$ and (x, y, c) is bad, then (3.31) follows directly from the definition of a bad program, and the fact that $\bar{c}_t = c^*$ for $t \geq 1$. This establishes the optimality of $(\bar{x}, \bar{y}, \bar{c})$.

Case (ii). Consider the pure accumulation program $(\hat{x}, \hat{y}, \hat{c})$ from y (recall example 3.1). We know that $\hat{y}_t \to K > x^*_\delta$ as $t \to \infty$. Let $\tau \geq 1$ be the first period for which $\hat{y}_\tau \geq x^*_\delta$. Then, clearly, $\bar{y}_t = \hat{y}_t = \hat{x}_t = \bar{x}_t$ and $\bar{c}_t = 0$ for $t = 0, \ldots, \tau - 1$; and $\bar{x}_t = x^*_\delta$ for $t \geq \tau$. Thus, for any $T > \tau$, we get

$$\sum_{t=0}^{T} \delta^t (c_t - \bar{c}_t) = \sum_{t=0}^{\tau-1} \delta^t \{[\delta f(x_t) - x_t] - [\delta f(\hat{x}_t) - \hat{x}_t]\} + \sum_{t=\tau}^{T-1} \delta^t \{[\delta f(x_t) - x_t] - [\delta f(x^*_\delta) - x^*_\delta]\} + \delta^T (x^*_\delta - x_T)$$

As noted in Case (i), for all $x \geq 0$, $[\delta f(x) - x] \leq [\delta f(x^*_\delta) - x^*_\delta]$. Also, for $t = 0, \ldots, \tau - 1$, $\delta f(x_t) - x_t \leq \delta f(\hat{x}_t) - \hat{x}_t$, since $[\delta f(x) - x]$ is increasing for $0 \leq x < x^*_\delta$ and $\hat{x}_t \geq x_t$ for $t = 0, \ldots, \tau - 1$. Thus, for $T > \tau$,

$$\sum_{t=0}^{T} \delta^t (c_t - \bar{c}_t) \leq \delta^T (x^*_\delta - x_T)$$

Now, repeating the argument used in case (i), (x, y, c) is optimal.//

4. On the Optimality of Competitive Programs

The notion of a competitive program has been central to the discussion of the role of prices in guiding intertemporal resource allocation.

Let us define a *competitive program* from y as a sequence (x, y, c, p) such that (x, y, c) is a program from y, and $p = (p_t)$ is a positive sequence of prices satisfying for all $t \in N$,

$$\text{(G)} \quad \delta^t u(c_t) - p_t c_t \geq \delta^t u(c) - p_t c \quad \text{for all } c \geq 0$$

$$\text{(M)} \quad p_{t+1} f(x_t) - p_t x_t \geq p_{t+1} f(x) - p_t x \quad \text{for all } x \geq 0$$

The second condition (M) was introduced earlier in our discussion of efficiency. The first condition (G) can be interpreted in terms of constrained utility maximization. One can think of the conditions (G) and (M) in terms of a separation of consumption and production decisions by means of a price system, the theme that pervades the masterly exposition of a Robinson Crusoe economy by Koopmans (1957). We refer to the sequence $p = (p_t)$ as a system of *competitive prices* supporting the program (x, y, c). [In the literature, these prices are also sometimes called »shadow» or »stimulating» prices].

Example 4.1:

Consider, for $0 < \delta \leq 1$, the program (x^*, y^*, c^*) defined by (GR). Define a sequence $p^* = (p^*_t)$ by $p^*_t = \delta^t u'(c_\delta^*)$ for $t \geq 0$. It is easy to check that (x^*, y^*, c^*, p^*) is a competitive program from y^*_δ.

Our interest in competitive programs is naturally due to the following basic result.

Proposition 4.1:

If $(\bar{x}, \bar{y}, \bar{c}, \bar{p})$ is a competitive program from $\bar{y} > 0$, then $(\bar{x}, \bar{y}, \bar{c})$ is optimal if

(IF) $\lim_{t\to\infty} \bar{p}_t \bar{x}_t = 0$ [*when* $0<\delta<1$]

(BCV) $\sup_{t\geq 0} \bar{p}_t \bar{x}_t < \infty$ [*when* $\delta = 1$]

Proof:

Let (x, y, c) be a program from any $y>0$. Using (G) and (M), one gets for any $T\geq 1$,

(4.1)

$$\sum_{t=0}^{T} \delta^t [u(c_t) - u(\bar{c}_t)] \leq \sum_{t=0}^{T} \bar{p}_t (c_t - \bar{c}_t)$$

$$= \sum_{t=0}^{T-1} \{[\bar{p}_{t+1} f(x_t) - \bar{p}_t x_t] - [\bar{p}_{t+1} f(\bar{x}_t) - \bar{p}_t \bar{x}_t]\}$$

$$+ \bar{p}_T(\bar{x}_T - x_T) + \bar{p}_0 (y - \bar{y})$$

$$\leq \bar{p}_T(\bar{x}_T - x_T) + \bar{p}_0 (y - \bar{y})$$

Consider, first, the case $0<\delta<1$. Using (4.1), if (x', y', c') is any program from $\bar{y}$, then

$$\sum_{t=0}^{T} \delta^t u(c'_t) - \sum_{t=0}^{T} \delta^t u(\bar{c}_t) \leq \bar{p}_T \bar{x}_T$$

The left-hand side expressions have limits (see Section III.d.1), and (IF) ensures that the limit of the right-hand side is zero. Hence,

$$\sum_{t=0}^{\infty} \delta^t u(c'_t) \leq \sum_{t=0}^{\infty} \delta^t u(\bar{c}_t)$$

establishing the optimality of $(\bar{x}, \bar{y}, \bar{c})$.

Consider, next, the case $\delta=1$. Using (4.1),

$$\sum_{t=0}^{T} [u(c^*) - u(\bar{c}_t)] \leq \bar{p}_T \bar{x}_T + \bar{p}_0 y^* \qquad \text{for all } T\geq 1$$

Using (BCV), it follows that $(\bar{x}, \bar{y}, \bar{c})$ is good, and so by Lemma 3.4, we have $(\bar{x}_t, \bar{y}_t, \bar{c}_t) \to (x^*, y^*, c^*)$ as $t\to\infty$. This fact, and (BCV) imply that there is some real number A, such that $\bar{p}_t \leq A$ for $t\geq 0$.

If (x', y', c') is any program from $\bar{y}$, then using (4.1), one gets for all $T\geq 1$

(4.2) $$\sum_{t=0}^{T} [u(c'_t) - u(\bar{c}_t)] \leq \bar{p}_T (\bar{x}_T - x'_T)$$

If (x', y', c') is good, then $x'_T \to x^*$ as $T\to\infty$. Also, $\bar{x}_T \to x^*$ as $T\to\infty$, and $0\leq \bar{p}_T \leq A$ for all T. Using these facts in (4.2) we get

(4.3) $$\limsup_{T\to\infty} \sum_{t=0}^{T} [u(c'_t) - u(\bar{c}_t)] \leq 0$$

If (x', y', c') is bad, then (4.3) follows directly from the definition of a bad program, and the fact that $(\bar{x}, \bar{y}, \bar{c})$ is a good program.//

Remark:

We have already encountered condition (IF) in our discussion of efficiency. The condition (BCV) is usually referred to in the literature as the »bounded capital value» condition.

The relationship between competitive and optimal programs can be explored further; specifically, it is of considerable interest to know that the converse of Proposition 4.1 is also true. We will demonstrate this only for the case of the »strictly concave» utility function [that is, when u satisfies (U.1), (U.2$^+$) and (U.3)]. In the more general case, the proof is considerably more involved.

We begin by noting the following characterization of competitive programs in the strictly concave case.

Lemma 4.1

Let (x, y, c) *be a program from* $y>0$. *There is a positive price sequence* (p) *satisfying* (G) *and* (M) *if and only if*

(i) $x_t>0$, $y_t>0$, $c_t>0$ *for* $t\geq 0$

(RE) (ii) $u'(c_t) = \delta u'(c_{t+1}) f'(x_t)$ *for* $t\geq 0$

Proof:

(Necessity) Using (U.3) and (G), $c_t>0$ for $t\geq 0$. Since $y_t \geq c_t$, we get $y_t>0$ for $t\geq 0$. Using (F.2), $x_t>0$ for $t\geq 0$. This establishes (i).

Using (G) and $c_t>0$, one obtains

$$p_t = \delta^t u'(c_t) \quad \text{for } t\geq 0$$

Using (M) and $x_t>0$, one obtains

$$p_{t+1} f'(x_t) = p_t \quad \text{for } t\geq 0$$

Combining the above two equations establishes (ii).

(Sufficiency) Let $p_t = \delta^t u'(c_t)$ for $t\geq 0$. This

defines a positive price sequence, using (i). Now, using (ii), one obtains

(4.4) $p_{t+1}f'(x_t)=p_t$ for $t\geq 0$

Given any $t\geq 0$, and $c\geq 0$ concavity of u leads to

$\delta^t[u(c)-u(c_t)]\leq \delta^t u'(c_t)(c-c_t)=p_t(c-c_t)$

so that (G) follows by transposing terms. Given any $t\geq 0$ and $x\geq 0$, concavity of f leads to

$p_{t+1}[f(x)-f(x_t)]\leq p_{t+1}f'(x_t)(x-x_t)=p_t(x-x_t)$

using (4.4). Thus (M) follows by transposing terms.//

Remark:

The condition (RE), known as the Ramsey-Euler condition, asserts the equality of the marginal product of input with the intertemporal marginal rate of substitution on the consumption side.

Now, we are in a position to establish a converse of Proposition 4.1.

Proposition 4.2

Suppose (x, y, c) *is an optimal program from* $\bar{y}>0$. *Then there is a positive price sequence* $p=(p_t)$ *satisfying* (G) *and* (M), and

(IF) $\lim_{t\to\infty} p_t x_t = 0$ [*when* $0<\delta<1$]

(BCV) $\sup_{t\geq 0} p_t x_t<\infty$ [*when* $\delta=1$]

Proof:

Since (x, y, c) is optimal from $\bar{y}>0$, $0<c_t<y_t$ for all $t\geq 0$, by using Lemma 3.7 (ii). Using (F.2), $x_t>0$ for $t\geq 0$.

Since (x, y, c) is optimal from $\bar{y}>0$, for each $t\geq 0$, x_t must maximize

$$W(x)\equiv\delta^t u(y_t-x)+\delta^{t+1}u(f(x)-x_{t+1})$$

among all x satisfying $0\leq x\leq y_t$ and $f(x)\geq x_{t+1}$. Since $c_t>0$ and $c_{t+1}>0$, the maximum is attained at an interior point. Thus, $W'(x_t)=0$, which can be written as

$$\delta^t u'(c_t)=\delta^{t+1}u'(c_{t+1})f'(x_t)$$

Thus (x, y, c) satisfies (i) and (ii) of Lemma 4.1. Consequently, there is a positive price sequence $p=(p_t)$ satisfying (G) and (M). In fact, using (G), and $c_t>0$, we have

$$p_t=\delta^t u'(c_t) \quad \text{for } t\geq 0$$

Using Lemma 3.7 (iii) we also get

(4.5) $p_t=\delta^t V'(y_t)$ for $t\geq 0$

Using the concavity of V and (4.5), we obtain for any $y>0$

$$\delta^t[V(y)-V(y_t)]\leq\delta^t V'(y_t)(y-y_t)=p_t(y-y_t)$$

Thus for all $t\geq 0$, and any $y>0$

(4.6) $\delta^t V(y)-p_t y\leq\delta^t V(y_t)-p_t y_t$

Choosing $y=(y_t/2)$ in (4.6), we get

(4.7) $(1/2)p_t y_t\leq\delta^t[V(y_t)-V(y_t/2)]$

If $0<\delta<1$, $V(y_t)\leq[u(K(\bar{y}))/(1-\delta)]$ and $V(y_t/2)\geq 0$ for $t\geq 0$ [see Section III.d.1]. Then (4.7) yields condition (IF), since $0\leq x_t\leq y_t$ for $t\geq 0$.

If $\delta=1$, (x, y, c) is good (using Lemma 3.3) and so $x_t\to x^*$, $y_t\to y^*$ and $c_t\to c^*$ as $t\to\infty$ (using Lemma 3.4). Thus, $p_t y_t\to u'(c^*)y^*$ as $t\to\infty$. This yields (BCV), since $0\leq x_t\leq y_t$ for $t\geq 0$.//

When one is interested in attaining the optimal program through an intertemporally decentralized, price-guided system of decision making, verification of the insignificant future (IF) or the bounded capital value condition (BCV) poses conceptual difficulties similar to those raised by Koopmans that we have recalled in the context of intertemporal efficiency. The problem of desiging a 'decentralizable rule' for myopically behaving firms or consumers continued to engage the attention of researchers (see Kurz and Starrett (1970), Starrett (1968)). However, significant progress towards designing 'decentralized' resource allocation mechanisms that can detect the long-run inefficiency or non-optimality of competitive programs through a sequence of period-by-period 'verifications' was achieved only recently. (Formal definitions of these concepts are taken up in Section VI). We shall now review some »possibility» results on verifying optimality of competitive programs reported in the *Symposium* (on intertemporal decentralization) in the *Journal of Economic Theory* (1988). We continue to focus on the case where the utility function is »strictly concave» [that is, u satisfies (U.1), ($U.2^+$) and (U.3)]. The particular period-by-period verification rule, which has figured prominently in this literature, can be approached in the following way.

An important monotonicity property of optimal programs (see Mitra (1979)) is the following: let (x, y, c) be optimal from $y>0$ and (x', y', c') be optimal from $y'>0$. If $y>y'$, one has:

(4.8) $x_t \geq x'_t$, $y_t \geqq y'_t$, $c_t \geqq c'_t$ for all $t \varepsilon N$

Given $0<\delta\leq 1$, the stationary program (x^*, y^*, c^*) defined in (GR) is optimal from y^*_δ. It follows that if (x, y, c) is optimal from y, $p^*_t = \delta^t u'(c^*_\delta)$, and $p_t = \delta^t u'(c_t)$, then

(S) $(p_t - p^*_t)(y_t - y^*_\delta) \leqq 0$ for all $t \varepsilon N$.

Indeed, one can show that the condition (S) *completely identifies* all competitive programs that are optimal.

Proposition 4.3

A competitive program (x, y, c, p) *from* $y>0$ *is optimal if and only if*

(S) $(p_t - p^*_t)(y_t - y^*_\delta) \leq 0$ *for all* $t \varepsilon N$

A sketch of an elementary proof [dealing with both the discounted and undiscounted cases in a single framework] is available in Majumdar (1987). An alternative characterization is provided by:

Proposition 4.4

A competitive program (x, y, c, p) *from* $y>0$ *is optimal if and only if*

(4.9) $x_{t+1} \lesseqqgtr x_t$ *whenever* $x_t \gtreqqless x^*_\delta$ *for all* $t \varepsilon N$

5. Role of Competitive Prices in Achieving Optimality: Some Further Results on the Strictly Concave Case

In our one good framework, the results of the previous section can be strengthened further. In this section we want to point out that there are alternatives to the condition (S) that we discussed earlier that do not involve the golden rule (or modified golden-rule) stock y^*_δ.

Lemma 5.1

Suppose (x, y, c, p) *is a competitive program from* $y \varepsilon (0, K)$.

(i) *If, for some* $s\geq 0$, *we have* $y_{s+1}>y_s$, *and* $x_s \geq x^*_\delta$, *then* $x_{t+1}>x_t$ *and* $y_{t+1}>y_t$ *for all* $t\geq s$.

(ii) *If, for some* $s\geq 0$, *we have* $y_{s+1}<y_s$ *and* $x_s \leq x^*_\delta$, *then* $x_{t+1}<x_t$ *and* $y_{t+1}<y_t$ *for all* $t\geq s$.

Proof:

We will prove (i). Suppose for some $s\geq 0$, $y_{s+1}>y_s$ and $x_s \geq x^*_\delta$. Then $\delta f'(x_s) \leq 1$, and using condition (RE) we get

$$u'(c_s) = \delta f'(x_s) u'(c_{s+1}) \leq u'(c_{s+1})$$

so that $c_s \geq c_{s+1}$. Thus $y_s - x_s \geq y_{s+1} - x_{s+1} > y_s - x_{s+1}$, using $y_{s+1}>y_s$. This yields $x_{s+1}>x_s$ and so $y_{s+2} = f(x_{s+1}) > f(x_s) = y_{s+1}$. Thus $x_{s+1} \geq x^*_\delta$ and $y_{s+2}>y_{s+1}$. This step can be repeated to get $x_{t+1}>x_t$ and $y_{t+1}>y_t$ for all $t\geq s$. The proof of (ii) is similar.//

Given a competitive program (x, y, c, p) from $y \varepsilon (0, K)$ define the *current value price sequence* $q=(q_t)$ by

$$q_t = (p_t/\delta^t) \quad \text{for } t \varepsilon N$$

Now, consider the following period-by-period verification rule:

(S') $(q_{t+1} - q_t)(y_{t+1} - y_t) \leq 0$ for $t \varepsilon N$

In contrast with the earlier condition (S), this condition (S') does not involve the explicit appearance of the golden-rule (or modified golden-rule) stock y^*_δ or prices p^*_t. We will show that a competitive program satisfies condition (S') if and only if it is optimal.

It is worth observing that a competitive program (x, y, c, p) for $y \varepsilon (0, K)$ satisfies (S') if and only if it satisfies

$$\text{(S'')} \quad (q_{t+1} - q_t)(x_{t+1} - x_t) \leq 0 \quad \text{for } t \varepsilon N$$

To see this, note that by condition (G) we get for $t \varepsilon N$

$$u(c_t) - q_t c_t \geq u(c_{t+1}) - q_t c_{t+1}$$
$$u(c_{t+1}) - q_{t+1} c_{t+1} \geq u(c_t) - q_{t+1} c_t$$

Adding the inequalities and transposing terms, we get

$$(5.1) \quad (q_{t+1} - q_t)(c_{t+1} - c_t) \leq 0 \quad \text{for } t \varepsilon N$$

Thus if (S'') is satisfied, adding the inequalities in (S'') and (5.1), one obtains (S').

To go in the other direction, note that by condition (M), we get for $t \varepsilon N$

$$\delta q_{t+1} f(x_t) - q_t x_t \geq \delta q_{t+1} f(x_{t+1}) - q_t x_{t+1}$$
$$\delta q_{t+2} f(x_{t+1}) - q_{t+1} x_{t+1} \geq \delta q_{t+2} f(x_t) - q_{t+1} x_t$$

Adding the inequalities and transposing terms, we get

$$(5.2) \quad \delta(q_{t+2} - q_{t+1})(y_{t+2} - y_{t+1}) \geq (q_{t+1} - q_t)(x_{t+1} - x_t) \quad \text{for } t \varepsilon N$$

Thus, if (S') is satisfied, it follows from (5.2) that (S'') is satisfied as well.

Lemma 5.2:

Suppose (x, y, c, p) *is a competitive program from* $y \varepsilon (0, K)$, *which satisfies* (S').

(i) *If, for some* $s \geq 0$, $x_s < x^*_\delta$, *then* $x_{s+1} \geq x_s$ *and* $y_{s+1} \geq y_s$.

(ii) *If, for some* $s \geq 0$, $x_s > x^*_\delta$, *then* $x_{s+1} \leq x_s$ *and* $y_{s+1} \leq y_s$.

Proof:

We will prove (i). Suppose, for some $s \geq 0$, $x_s < x^*_\delta$. Then $\delta f'(x_s) > 1$. Using condition (RE), $q_{s+1} = [q_s/\delta f'(x_s)] < q_s$. Using this in (S'), one obtains $y_{s+1} \geq y_s$; using this in (S''), one obtains $x_{s+1} \geq x_s$. The proof of (ii) is similar.//

Using Lemmas 5.1 and 5.2 one can show that a competitive program satisfying (S') must converge to the golden rule (or modified golden-rule).

Proposition 5.1:

Suppose (x, y, c, p) *is a competitive program from* $y \varepsilon (0, K)$, *which satisfies* (S'). *Then* $(x_t, y_t, c_t) \to (x^*_\delta, y^*_\delta, c^*_\delta)$ *as* $t \to \infty$.

Proof:

Suppose $y < y^*_\delta$. We claim that

$$(5.3) \quad x_t < x^*_\delta \quad \text{for } t \varepsilon N$$

For if (5.3) is violated, let s be the first period for which $x_s \geq x^*_\delta$. If $s = 0$, we have $y_{s+1} = f(x_s) \geq f(x^*_\delta) = y^*_\delta > y = y_s$. If $s \geq 1$, we have $x_{s-1} < x^*_\delta$, and so $y_{s+1} = f(x_s) \geq f(x^*_\delta) > f(x_{s-1}) = y_s$. Thus, in either case, $y_{s+1} > y_s$. Applying Lemma 5.1, $x_{s+1} > x_s \geq x^*_\delta$ and $y_{s+2} > y_{s+1}$, which contradicts Lemma 5.2. This establishes our claim.

Using (5.3) and Lemma 5.2, we can conclude that $x_{t+1} \geq x_t$ and $x_t < x^*_\delta$ for $t \geq 0$. Thus x_t converges to some $\bar{x}$ satisfying $0 < \bar{x} \leq x^*_\delta < K$, and $[f(x_{t-1}) - x_t]$ converges to $[f(\bar{x}) - \bar{x}] > 0$. Using condition (RE), we get $\delta f'(\bar{x}) = 1$, so that $\bar{x} = x^*_\delta$. Thus $x_t \to x^*_\delta$ as $t \to \infty$; also, $y_{t+1} = f(x_t) \to f(x^*_\delta) = y^*_\delta$ as $t \to \infty$; and $c_t = y_t - x_t \to (y^*_\delta - x^*_\delta) = c^*_\delta$ as $t \to \infty$. When $y \geq y^*_\delta$, the proof is similar.//

Our main result can now be stated as follows:

Proposition 5.2

Suppose (x, y, c, p) *is a competitive program from* $y \varepsilon (0, K)$ *which satisfies* (S'). *Then* (x, y, c) *is an optimal program from* y.

Proof:

If $\delta = 1$, Proposition 5.1 implies that $p_t \equiv u'(c_t) \to u'(c^*) = p^*$ as $t \to \infty$. Also $x_t \to x^*$ as $t \to \infty$. Thus condition (BCV) is clearly satisfied, and (x, y, c) is optimal by Proposition 4.1.

If $0 < \delta < 1$, Proposition 5.1 implies that $q_t = u'(c_t) \to u'(c^*_\delta)$ as $t \to \infty$, and so $p_t = \delta^t q_t \to 0$ as $t \to \infty$. Also $x_t \to x^*_\delta$, so condition (IF) is clearly satisfied, and (x, y, c) is optimal by Proposition 4.1.//

A converse of Proposition 5.2 can also be established, and is presented below in the next result.

Proposition 5.3

Suppose (x, y, c, p) *is a competitive pro-*

gram from y ε (0, K), *such that* (x, y, c) *is optimal from* y. *Then* (x, y, c, p) *satisfies* (S').

Proof:

Using Lemma 4.1, $x_t>0$, $y_t>0$, $c_t>0$ for t ε N. Thus, condition (G) yields $q_t=u'(c_t)$ for t ε N. Using Lemma 3.7, we also have $q_t=V'(y_t)$ for t ε N.

Since V is concave (by Lemma 3.6), we have for any y>0, and t ε N,

$$V(y)-V(y_t)\leq V'(y_t)(y-y_t)=q_t(y-y_t)$$

Transposing terms, we get for any y>0, and t ε N

$$(5.4)\quad V(y_t)-q_ty_t\geq V(y)-q_ty$$

Pick any s≥0. Then using (5.4) for t=s, and $y=y_{s+1}$,

$$(5.5)\quad V(y_s)-q_sy_s\geq V(y_{s+1})-q_sy_{s+1}$$

Using (5.4) for t=s+1, and $y=y_s$, we have

$$(5.6)\quad V(y_{s+1})-q_{s+1}y_{s+1}\geq V(y_s)-q_{s+1}y_s$$

Adding (5.5) and (5.6) and transposing terms,

$$(q_{s+1}-q_s)(y_{s+1}-y_s)\leq 0$$

Since s≥0 was arbitrarily picked, this establishes condition (S').//

6. *Intertemporal Decentralized Mechanisms*

The framework introduced in Section II, in our discussion of decentralized resource allocation mechanisms, is a flexible one: it can be formally adapted to the intertemporal context of Section III, with some changes, as follows. One can identify the set of agents, I, to be the consumers and/or producers in the intertemporal economy over the infinite horizon. The environment of each producer can be specified to be the set of production functions introduced in Section III b. The initial producer's environment can contain information about the initial stock of the economy. The environment of each consumer can be specified to be the set of utility functions introduced in Section III d. The space of allocations, A, can be specified to be the space in which programs will belong, namely S_+^3.

»Intertemporal decentralized mechanisms» can be thought of as decentralized mechanisms (in the sense of Section II) applied to the above intertemporal framework. The social goal correspondence can be defined to be the set of efficient or optimal programs from the given initial stock of the economy.

The important question then is whether one can devise suitable intertemporal decentralized mechanisms which realize the above social goals. We now present some examples to illustrate how this question can be answered, by using our results on efficient and optimal programs developed in the previous three sections. The examples also indicate precisely how intertemporal decentralized mechanisms can be constructed, an aspect we have deliberately avoided discussing in detail so far, in view of the unavoidable technicalities involved.

Example 6.1:

The set of agents, I={t ε N}. We think of agent t as a »producer» living at period t. The set of environments, E, is defined as follows. Let E_t={f: f satisfies (F.1)−(F.4)} for t≥1; let E_0={(y, f) : satisfies (F.1)−(F.4) and y ε $\mathbb{R}_{++}$}. Now E is defined as {y, f^∞): y ε $\mathbb{R}_{++}$ and f satisfies (F.1)−(F.4}. Thus $E\subset\prod_{t=0}^{\infty}E_t$, and our given environment is a *stationary* one. Define the space of allocations, A to be S_+^3.

The social goal correspondence is the mapping Q: E → A, defined by Q(e)={(x, y, c) : (x, y, c) is an efficient program when the environment is e}.

Consider a mechanism π=(M, ψ, H) defined as follows. First, the message space, M, is $S_+^3\times\hat{S}$, with generic element written as (x, y, c, p). Next, the equilibrium correspondence ψ is defined as follows. For e_0 in E_0 let $\psi_0(e_0)$={(x, y, c, p) in M: $x_0+c_0=y_0$, $y_0=y$, $y_1=f(x_0)$ and $p_1f(x_0)-p_0x_0\geq p_1f(x)-p_0x$ for all x≥0}; for e_t in E_t, let $\psi_t(e_t)$={(x, y, c, p) in M: $x_t+c_t=y_t$, $y_{t+1}=f(x_t)$ and $p_{t+1}f(x_t)-p_tx_t\geq p_{t+1}f(x)-p_tx$ for all x≥0} for t≥1. Now, given any e in E, ψ(e) is defined as $\bigcap_{t=0}^{\infty}\psi_t(e_t)$. Since there exists a program from each y>0, Proposition 3.1 ensures that ψ is non-empty valued for each e in E, as required. Finally, the outcome function, H, is defined as follows. For $m = (x, y, c, p)$ in M, $H(m) = (x, y, c)$. Thus, H is a projection mapping from M to A. The performance correspondence, $\phi(e)$, is the set of programs from y, by Proposition 3.1.

Clearly, the mechanism, π, is privacy preserving by definition. It is unbiased by Proposition 3.1, but in view of Example 3.1, it does not realize the social goal of efficiency.

The example emphasizes the point that the conditional of intertemporal profit maximization (see (M) in Section III) does not ensure efficiency. In our framework, it does not restrict the class of programs at all.

Example 6.2

The set of agents, $I = \{(t_R, t_c) : t \in N\}$. We think of agent t_R as a »producer» Robinson living in period t, and t_c as a »consumer» Crusoe living in period t. The set of environments, E, is defined as follows. Let $E_{t_R} = \{(f, \delta)$: f satisfies (F.1)–(F.4), and $\delta \in (0, 1]\}$ for $t \geq 1$; let $E_{0_R} = \{y, f, \delta)$: $y \in \mathbb{R}_{++}$, f satisfies (F.1)–(F.4), and $\delta \in (0, 1]$. Let $E_{t_c} = \{u: \mathbb{R}_+ \to \mathbb{R}$ such that $u(c) = c$ for $c \in \mathbb{R}_+\}$ for $t \geq 0$. Now, E is defined as $\{(y, f^\infty, \delta^\infty, u^\infty)$: $y \in \mathbb{R}_{++}$, f satisfies (F.1)–(F.4), $\delta \in (0, 11]$, and u: $\mathbb{R}_+ \to \mathbb{R}$ such that $u(c) = c$ for $c \in \mathbb{R}_+\}$. Thus, $E \subset \prod_{t=0}^{\infty} E_{t_R} \times \prod_{t=0}^{\infty} E_{t_c}$, and the given environment is a stationary one. The space of allocations, A, is S_+^3.

The social goal correspondence is the mapping $Q : E \to A$, defined by $Q(e) = \{(x, y, c)$: (x, y, c) is an optimal program, when the environment is e}.

Consider a mechanism $\pi = (M, \psi, H)$ defined as follows. First, the message space, M, is $S_+^3 \times \mathbb{R}_{++}$, with generic element written as (x, y, c, x_δ^*). Next, the equilibrium correspondence ψ is defined as follows. For e_{0_R} in E_{0_R}, let $\psi_{0_R}(e_{0_R}) = \{(x, y, c, x_\delta^*)$ in $M : x_0 + c_0 = y_0$, $y_0 = y$, $y_1 = f(x_0)$, $\delta f'(x_\delta^*) = 1$, and $c_0 = \max(y_0 - x_\delta^*, 0)\}$; For e_{t_R} in E_{t_R}, let $\psi_{t_R}(e_{t_R}) = \{(x, y, c, x_\delta^*)$ in $M : x_t + c_t = y_t$, $y_{t+1} = f(x_t)$, $\delta f'(x_\delta^*) = 1$, and $c_t = \max(y_t - x_\delta^*, 0)\}$ for $t \geq 1$. For e_{t_c} in E_{t_c}, let $\psi_t(e_{t_c}) = M$ for $t \geq 0$. Now, define for e in E, $\psi(e) = [\bigcap_{t=0}^{\infty} \psi_{t_R}(e_{t_R})] \cap [\bigcap_{t=0}^{\infty} \psi_{t_c}(e_{t_c})]$. Since there is a golden-rule (or modified golden-rule) input (see Section III.b) and there is an optimal program from every $y > 0$, (see Section III.d), Lemma 3.8 ensures that ψ is non-empty valued for each e in E, as required. Finally, the outcome function, H, is defined, for each $m = (x, y, c, x_\delta^*)$ by $H(m) = (x, y, c)$. Thus, H is a projection mapping from M to A. The performance correspondence, $\phi(e)$, is the unique optimal program from y (using Lemma 3.8).

The mechanism, π, is privacy preserving by definition, and it realizes the social goal of optimality.

A remark regarding the equilibrium correspondence is worth making. The key idea is that the optimal consumption policy function, g, for this social goal can be explicitly written as

$$g(y) = \max(y - x_\delta^*, 0) \quad \text{for all } y > 0$$

where x_δ^* is the golden-rule ($\delta = 1$) or modified golden-rule ($0 < \delta < 1$) input (Lemma 3.8). Thus, to obtain a mechanism which realizes the social goal, one can ask the producers to verify whether $c_t = \max(y_t - x_\delta^*, 0)$ for each period. Since this involves x_δ^*, which is defined as the solution to $\delta f'(x_\delta^*) = 1$, one can ask producers in each period to calculate it, using their knowledge of δ and f, and then use it in the above verification. Notice that consumers are not asked to verify anything; their role (given the linear utility function) is indeed a limited one in this mechanism.

Example 6.3

The set of agents, I, the space of allocations, A, and the sets, E_{t_R} (for $t \geq 0$), are defined as in Example 6.2. Let $E_{t_c} = \{u : \mathbb{R}_+ \to \mathbb{R}$, such that u satisfies (U.1), (U.2$^+$) and (U.3)$\}$. Now, E is defined as $\{(y, f^\infty, \delta^\infty, u^\infty)$: $y \in \mathbb{R}_{++}$, f satisfies (F.1)–(F.4), $\delta \in (0, 1)$, and $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies (U.1), (U.2$^+$) and (U.3)$\}$. The social goal is optimality.

Consider a mechanism $\pi = (M, \psi, H)$ defined as follows. First, the message space, M, is $S_{++}^4 \times \mathbb{R}_{++}^4$, with generic element written as $(x, y, c, q, x_\delta^*, y_\delta^*, c_\delta^*, q^*)$. Next, let $\psi_{0_R}(e_{0_R}) = \{(x, y, c, q, x_\delta^*, y_\delta^*, c_\delta^*, q^*)$ in $M : x_0 + c_0 = y_0$, $y_0 = y$, $\delta q_1 f'(x_0) = q_0$, $\delta f'(x_\delta^*) = 1$, $y_\delta^* = f(x_\delta^*)$, $x_\delta^* + c_\delta^* = y_\delta^*$, and $(q_0 - q^*)(y_0 - y_\delta^*) \leq 0\}$. Further, let $\psi_{t_R}(e_{t_R}) = \{x, y, c, q, x_\delta^*, y_\delta^*, c_\delta^*, q^*)$ in $M : x_t + c_t = y_t$, $y_{t+1} = f(x_t)$, $\delta q_{t+1} f'(x_t) = q_t$, $\delta f'(x_\delta^*) = 1$, $y_\delta^* = f(x_\delta^*)$, $x_\delta^* + c_\delta^* = y_\delta^*$, and $(q_t - q^*)(y_t - y_\delta^*) \leq 0\}$ for $t \geq 1$. For $t \geq 0$, $\psi_{t_c}(e_{t_c}) =$

$\{(x, y, c, q, x^*_\delta, y^*_\delta, c^*_\delta, q^*)$ in $M : u'(c_t) = q_t$, $u'(c^*_\delta) = q^*\}$. Now define for e in E, $\psi(e) = [\bigcap_{t=0}^{\infty} \psi_{t_R}(e_{t_R})] \cap [\bigcap_{t=0}^{\infty} \psi_{t_c}(e_{t_c})]$ an optimal program (x, y, c) from every $y > 0$, Propositions 4.2 and 4.3 yield (p) such that (G), (M) and (S) are satisfied. Defining (q) by $q_t = (p_t/\delta^t)$ for $t \geq 0$, it can be checked using Lemma 4.1 that $u'(c_t) = q_t$ for $t \geq 0$ and $\delta q_t f'(x_{t-1}) = q_{t-1}$ for $t \geq 1$. Thus, $\psi(e)$ is non-empty valued as required.

The outcome function, H, is defined for each $m = (x, y, c, q, x^*_\delta, y^*_\delta, c^*_\delta, q^*)$ by $H(m) = (x, y, c)$. Thus, H maps from M to A. The performance correspondence, $\phi(e)$, is the unique optimal program from y, using Lemma 4.1 and Porpositions 4.2 and 4.3.

The mechanism, π, is privacy-preserving by definition, and it realizes the social goal of optimality.

The key idea used here is that optimal programs can be characterized by conditions (G), (M) and (S) (besides feasibility). Thus, to obtain a mechanism which realizes the social goal, one can ask consumers to check (G) and producers to check (M) and (S) and feasibility. Now (S) itself involves the golden-rule (or modified golden-rule) stock, and price. While one cannot aks producers by themselves to calculate these (as we did in Example 6.2), we can obtain them through a privacy-preserving mechanism involving consumers and producers [by asking producers to check $\delta f'(x^*_\delta) = 1$, $y^*_\delta = f(x^*_\delta)$ and $x^*_\delta + c^*_\delta = y^*_\delta$, and asking consumers to check $u'(c^*_\delta) = q^*$]. The magnitudes can then be used to check condition (S) for each period.

Example 6.4

The set of agents and space of allocations are as described in Example 6.3. To define the set of environments, let $\xi > 0$ be a fixed real number, and $\bar{k}$ be another fixed real number satisfying $0 < \bar{k} < \xi$. Define $\mathscr{F} = \{f : f$ satisfies (F.1) – (F.4) and $f(\bar{k}) \geq \bar{k}$, $f(\xi) < \xi$, $f'(\bar{k}) < 1\}$. Let $E_{t_R} = \{(f, \delta) : f \in \mathscr{F}$ and $\delta \in (0, 1]\}$ for $t \geq 1$; let $E_{o_R} = \{(y, f, \delta) : y \in (0, \bar{k})$, $f \in \mathscr{F}$ and $\delta \in (0, 1]\}$. Further, let $E_{t_c} = \{u : \mathbb{R}_+ \to \mathbb{R}$ such that u satisfies (U.1), (U.2$^+$) and (U.3)$\}$. Now, E is defined as $\{(y, f^\infty, \delta^\infty, u^\infty) : y \in (0, \bar{k})$, $f \in \mathscr{F}$, $\delta \in (0, 1]$, and u: $\mathbb{R}_+ \to \mathbb{R}$ satisfies (U.1), (U.2$^+$) and (U.3)$\}$. The social goal is optimality.

Consider a mechanism $\pi = (M, \psi, H)$ defined as follows. The message space is $M = S^4_{++}$ with generic element written as (x, y, c, q). Next, let $\psi_{o_R}(e_{o_R}) = \{(x, y, c, q)$ in $M : x_o + c_o = y_o$, $y_o = y$, $y_1 = f(x_o)$, $\delta q_1 f'(x_o) = q_o$, and $(q_1 - q_o)(y_1 - y_o) \leq 0\}$. Further, let $\psi_{t_R}(e_{t_R}) = \{(x, y, c, q)$ in $M : x_t + c_t = y_t$, $y_{t+1} = f(x_t)$, $\delta q_{t+1} f'(x_t) = q_t$, and $(q_{t+1} - q_t)(y_{t+1} - y_t) \leq 0\}$ for $t \geq 1$. For $t \geq 0$, $\psi_{t_c}(e_{t_c}) = \{(x, y, c, q)$ in $M : u'(c_t) = q_t\}$. Now, define $\psi(e) = [\bigcap_{t=0}^{\infty} \psi_{t_R}(e_{t_R})] \cap [\bigcap_{t=0}^{\infty} \psi_{t_c}(e_{t_c})]$.

Since there is an optimal program (x, y, c) from $y \in (0, \bar{k})$, Proposition 4.2 yields (p) such that (G), (M) are satisfied. Defining (q) by $q_t = (p_t/\delta^t)$ for $t \geq 0$, Lemma 4.1 can be used to check that $u'(c_t) = q_t$ for $t \geq 0$ and $\delta q_t f'(x_{t-1}) = q_{t-1}$ for $t \geq 1$. Proposition 5.3 ensures that condition (S') is satisfied. Thus, $\psi(e)$ is non-empty valued as required.

The outcome function, H, is defined for each $m = (x, y, c, q)$ by $H(m) = (x, y, c)$. Thus H maps M to A. The performance correspondence, $\phi(e)$, is the unique optimal program from y, using Lemma 4.1 and Propositions 4.2 and 5.2.

The mechanism, π, is privacy preserving by definition, and it realizes the social goal of optimality.

Since condition (S') does not involve the golden-rule (or modified golden-rule) stock and price [unlike condition (S)], the mechanism discussed in this example can operate with a message space of smaller dimension compared to the mechanism used in Example 6.3. Further, agents have to verify fewer rules in the operation of this mechanism compared to the one used in Example 6.3. On the other hand, if we consider the set of environments for which we have been able to demonstrate that each mechanism realizes the social goal of optimality, we find that the set in this example is smaller than the set in Example 6.3.

7. *Intertemporal Mechanisms under Incomplete Information: An Impossibility Result*

In the previous section, we showed how the framework of decentralized resource allocation mechanisms can be formally adapted to the intertemporal context. However, once one enquires how such mechanisms would operate over an infinite horizon, it appears that the »verification scenario» would be a rather implausible one.

Consider, for instance, the mechanism in Example 6.4. Let us, in fact, consider the fortuitous case in which an *equilibrium* message is proposed. All agents from now to infinity have to verify that the message is indeed an equilibrium one. But, at the current date, not all agents are present — many are yet to be born. For an infinite-horizon economy, waiting till all the verification is complete before implementing the equilibrium allocation is really »... a prescription for economic paralysis rather than a realistic model for economic behavior...» (Hurwicz and Weinberger (1990, p. 317)).

To resolve this difficulty, it is clear that we have to restrict drastically the class of mechanisms which are acceptable, in the intertemporal context, as reflecting realistic economic behavior. This line of reasoning has led to research basically along two directions, the first of which (an impossibility result for non-stationary environment) we discuss in this section, leaving the second (a possibility result for a stationary environment) for the next section.

In order to discuss formally the first line of enquiry, let $\xi > 0$ be a real number, and define $\mathscr{F} = \{f : f$ satisfies (F.1)–(F.4), and $f(x) \leq x$ for $x \geq \xi\}$. We denote a sequence $(f_o, f_1, f_2, \ldots)$, where $f_t \in \mathscr{F}$ for $t \geq 0$, by f. Define $\mathcal{U} = \{u : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1), (U.2$^+$) and (U.3)$\}$.

Consider, now, an intertemporal economy in which the set of agents, $I = \{t \in N\}$, the space of allocations, A, is $\mathbb{R}^3_+$, and the set of environments, E, is $\{(f, u, \delta, y) : f_t \in \mathscr{F}$ for $t \geq 0$, $u \in \mathcal{U}$, $\delta \in (0, 1)$ and $y \in \mathbb{R}_{++}\}$. In particular, then, one is allowing for *changing* technology in the set of environments. Given e in E a *program* (x, y, c) is a sequence satisfying

$$y_o = y,\ x_t + c_t = y_t,\ y_{t+1} = f_t(x_t),$$
$$(x_t,\ c_t) \geq 0 \text{ for } t \geq 0$$

An optimal program is defined as before [see (3.14)]. Given our definition of $\mathscr{F}$, it is routine to check, following the method of proof of Proposition 3.3, that there is a unique optimal program, given an environment e in E.

Let us focus on a rather limited social goal. The social goal function $Q : E \to A$ is defined as follows. For $e \in E$,

$$Q(e) = \{x_o,\ y_o\ c_o) : (x, y, c)$$
$$\text{is the optimal program, given e}\}$$

Thus, the social goal is simply to attain the optimal allocation *in the initial period.*

It can be shown that the social goal is »sensitive» to a change in the environment. This can be formulated as follows. Given any production function, $f \in \mathscr{F}$, we can define $F : \mathbb{R}_+ \to \mathbb{R}_+$ by $F(x) = [f(x)]^\alpha$ for some $0 < \alpha < 1$; then $F \in \mathscr{F}$. Given any integer $\tau \geq 1$, one can then define $f(\tau) = (f_0(\tau), f_1(\tau), f_2(\tau), \ldots)$ by $f_t(\tau) = f$ for $t = 0, \ldots, \tau$ and $f_t(\tau) = F$ for $t > \tau$.

Proposition 7.1

Let $\tau \geq 1$ *be any integer. Given* $f \in \mathscr{F}$, $u \in \mathcal{U}$, *and* $0 < \delta < 1$, *there is* $y > 0$ *such that for the environments* $e \equiv (f^\infty, u, \delta, y)$ *and* $e(\tau) \equiv (f(\tau), u, \delta, y)$ *in* E, $Q(e) \neq Q(e(\tau))$.

We are now interested in realizing this social goal by constructing a suitable mechanism *for the initial period.* Unlike the intertemporal mechanisms of the previous section, we restrict this mechanism by requiring that its equilibrium message and outcome be *completely determined by the agent living in the initial period.* Regarding the agent living in the initial period, we suppose realistically that he has less than complete information about the environment. Specifically, there is a positive integer T such that the production functions, f_t, for $t > T$ are not known to him. Formally, given f, denote $(f_1, \ldots, f_T)$ by f^T. Now define the environment of agent zero, for e in E, by $E_o(e) = \{f^T, u, \delta, y) : e = (f, u, \delta, y)\}$, and $E_o = E_o(E)$.

Notice how much information we are allowing agent zero to possess. He knows the utility function, the discount factor, and the production functions up to some finite T. Thus one is not insisting on a separation of information between »consumers» and »producers». We *are* insisting that the agent does not know e *completely*, when e happens to be the actual environment, and our definition of $E_o(e)$ captures this in a minimally restrictive way.

Let us represent a mechanism for the initial period by $\pi_o = (M_o, \psi_o, H_o)$, where ψ_o is a non-empty valued mapping from E_o to M_o, and H_o is a mapping from M_o to A. We are now in a position to state a result on the impossibility of realizing the social goal by any mechanism. The idea is essentially simple. If we consider two environments which differ only in their production functions beyond T, the agent living in period zero cannot know about the difference. If a mechanism were to realize the social goal for each environment, then the social goal function must yield the same period zero allocation for both environments. But this contradicts the result established in Proposition 7.1 about the sensitivity of the social goal function to a change in environment.

Proposition 7.2:

There is no mechanism which can realize the social goal.

Proof:

Suppose a mechanism $\pi_o = (M_o, \psi_o, H_o)$ realizes the social goal. Let $\tau > T$ be any integer. Given $f \in \mathscr{F}$, $u \in \mathcal{U}$, and $0 < \delta < 1$, one can find (by Proposition 7.1) $y > 0$, such that for the environments $e \equiv (f^\infty, u, \delta, y)$ and $e(\tau) \equiv (f(\tau), u, \delta, y)$, $Q(e) \neq Q(e(\tau))$.

Now, since π_o realizes Q, and Q is a function, the performance correspondence ϕ is a function satisfying

$$\phi(E_o(e)) \quad = Q(e)$$

$$\phi(E_o(e(\tau))) = Q(e(\tau))$$

But since $E_o(e) = E_o(e(\tau))$, the above equalities imply $Q(e) = Q(e(\tau))$, a contradiction.//

The above discussion gives a flavor of the impossibility results which can be obtained when agents have incomplete information about the environment, and equilibrium allocations in a period are decided by the currently living agents. For a more exhaustive treatment, the reader is encouraged to consult Hurwicz and Majumdar (1988).

8. *Decentralized Evolutionary Mechanisms*

We have indicated in the previous section the kind of restrictions we need to place on mechanisms in order that they be plausible in the intertemporal context. In this section, we develop this theme more completely by introducing the notion of »decentralized evolutionary mechanisms».

What we would like to formalize now is the notion of a *sequence of mechanisms, one for each time period, such that only the agents living at a given date decide the equilibrium allocation of the mechanism at that date.* Further, this equilibrium allocation is actually carried out at that date, and is therefore independent of the opinions (verifications) of future agents.

In formalizing the above idea, a question arises regarding the treatment of the »initial stock». Notice that in our discussion of intertemporal mechanisms in Section VI and in constructing a mechanism for the initial period in Section VII, the initial stock was treated as part of the environment of the producer in the initial period. When we consider the mechanism for the date $t = 1$, its »initial stock» (y_1) is *determined by the equilibrium outcome of the mechanism at date* $t = 0$. Thus, the difficulty of treating it as part of the environment of the producer at date $t = 1$ is that then the environment at $t = 1$ becomes dependent on the mechanism used at date $t = 0$. The social goal correspondence, which is defined on the environment of the intertemporal economy, in turn, becomes dependent on the mechanism used at date $t = 0$. Thus, there is no longer a meaningful way in which the performance of a mechanism can be judged in terms of realization of social goals, since the latter concept is now *dependent* on the former one.

There is no question that for any sensible operation of the mechanism at date $t = 1$, the agents at date $t = 1$ must know their »initial stock», y_1. Thus, what seems to be called for is to treat y_1 as something to be determined at date $t = 0$ (by the mechanism operating in the initial period), but as something which is »common knowledge» (as a historically recorded fact) at date $t = 1$. The most convenient way to do this is to treat stock as a »state variable», and include in our description of mechanisms a suitable »state space».

We now proceed to formalize the above notions by defining an evolutionary mechanism. This is followed by an informal discussion of how much a mechanism is supposed to operate.

We consider the following objects to be given: a *set of agents* $I = \{t_c, t_R) : t \in N\}$, a *set of environments*, $E \subset \prod_{t=0}^{\infty} (E_{t_c} \times E_{t_R})$, a *space of allocations* $A = \prod_{t=0}^{\infty} A_t$, and a *state space* $S = \prod_{t=0}^{\infty} S_t$. We consider A_t to be a subset of a finite dimensional real space $\mathbb{R}^n$, and S_t to be a subset of a finite dimensional real space $\mathbb{R}^q$, for $t \in N$.

An *evolutionary mechanism* is a sequence $\pi = (M, \psi, H)$ where:

(a) M_t, the message space in period t, is a subset of a finite-dimensional real space, denoted by $\mathbb{R}^m$;
(b) ψ_t, the *equilibrium function* in period t, is a non-empty mapping from $E_t \times S_t$ to M_t.
(c) $H_t \equiv (H_t^1, H_t^2)$, the *outcome function* in period t, is a mapping from M_t to $\mathbb{R}^q \times \mathbb{R}^\lambda$, such that $H_t(m_t) \in S_t \times A_t$ if $m_t = \psi_t(e_t, s_t)$.

Given π, the *performance function in period t*, ϕ_t, is defined for each $(e_t, s_t) \in E_t \times S_t$ by ϕ_t by $\phi_t(e_t, s_t) = \{H_t(m_t) : m_t = \psi_t(e_t, s_t)\}$.

Thus, given $e \in E$ and $s \in S_o$, a mechanism π generates a *state sequence* s by

$$s_o = s, \; s_{t+1} = \phi_t^1(e_t, s_t) \quad \text{for } t \in N$$

and an *allocation* sequence a by

$$a_t = \phi_t^2(e_t, s_t) \quad \text{for } t \in N$$

Given π, the *performance function* ϕ from $E \times S_o$ to $S \times A$ is defined by $\phi(e, s) = \{(s, a) :$ s is the state sequence and a is the allocation sequence generated by $\pi\}$.

The above definition of an evolutionary mechanism is similar in spirit to the notion of an »evolutionary process» introduced by Hurwicz and Weinberger (1990, p. 317). However, in incorporating explicitly the state space and the state variable in defining the mechanism, our definition is somewhat different from theirs.

The dimensionality restriction on the message space reflects the notion that transmission and usage of information is costly and agents can process only a finite amount of information in each period.

The evolutionary mechanism π is *privacy preserving* or *decentralized* if there exist equilibrium correspondences $\psi_{t_c} : E_{t_c} \times S_t \to M_t$ and $\psi_{t_R} : E_{t_R} \times S_t \to M_t$ for all $t \in N$ such that

$$\psi_t(e_{t_c}, e_{t_R}, S_t) = \psi_{t_c}(e_{t_c}, S_t) \cap \psi_{t_R}(e_{t_R}, s_t)$$

for all (e_{t_c}, e_{t_R}, s_t) in $E_{t_c} \times E_{t_R} \times S_t$.

To see how this mechanism operates, consider that an environment, e_o, and a state, s_o, for period zero, are given. The environment would describe the preference of consumers and technological possibilities of producers in period zero. The state would describe the stocks of various goods available at the beginning of period zero.

A message m_o is proposed to the agents in period zero. The agents, knowing their respective environments, e_{o_R} and e_{o_c}, and the state, s_o (which is »common knowledge»)), check whether $m_o = \psi_{o_c}(e_{o_c}, s_o)$ and $m_o = \psi_{o_R}(e_{o_R}, s_o)$. If it is, then m_o is the equilibrium message, and the outcome function H_0^1 specifies a state s_1 in s_1, and H_0^2 specifies an allocation a_o in A_o consisting of consumption/investment decisions. This state, allocation pair is the equilibrium outcome of period zero. It is to be understood that the equilibrium outcome is actually carried out in period zero, and the state s_1 is actually attained (at the beginning of period 1). The above process is then repeated for $t = 1, 2, 3, \ldots$.

A *social goal correspondence* is a mapping Q from $E \times S_o$ to $S \times A$. The mechanism, π, *realizes* Q if for each (e, s_o) in $E \times S_o$, $\phi(e, s_o)$ belongs to $Q(e, s_o)$.

We now examine how the above concepts can be applied to our simple one-good intertemporal framework. We note, right away, that in view of the impossibility result discussed in Section VII, there does not seem to be any hope of realizing optimality in a non-stationary environment. Thus, we confine our attention to stationary environments in what follows. Define a class of production functions, $\mathcal{F}$, and a class of utility functions, $\mathcal{U}$ as in Example 6.4.

Consider a framework in which $E_{t_R} = \{f, \delta) : f \in \mathcal{F}$ and $0 < \delta < 1\}$, $E_{t_c} = \{u : u \in \mathcal{U}\}$, and $E = \{(u^\infty, f^\infty, \delta^\infty) : u \in \mathcal{U}, f \in \mathcal{F}$ and $0 < \delta < 1\}$. Further, let $A_t = \mathbb{R}_+^2$ and $S_t = 0, \bar{k})$ for $t \in N$. Finally, define $Q(e, y) = \{(x, y, c) : (x, y, c)$ is the optimal program, given $(e, y)\}$. The result of Hurwicz and Weinberger (1990) would indicate that there is no decentralized evolutionary mechanism which realizes the social goal. [Our statement is deliberately a qualified one, since the class of environments considered by Hurwicz-Weinberger is not exactly E, and also because they impose some »regularity» conditions on the evolutionary mechanisms which are allowed for.]

One may ask, in view of the negative result just stated, whether there is *any* decentralized evolutionary mechanism whose outcomes have some interesting normative properties. This is the main theme of Bala, Majumdar and Mitra (1990), and their result can be described as follows.

Consider a framework in which $E_{t_R} = \{(f, \delta) : f \varepsilon \mathscr{I} \text{ and } \delta = 1\}$, $E_{t_c} = \{u : u \varepsilon \mho\}$, and $E = \{(u^\infty, f^\infty, \delta^\infty) : u \varepsilon \mho, f \varepsilon \mathscr{I} \text{ and } \delta = 1\}$. Further, let $A_t = \mathbb{R}^2_+$ and $S_t = (0, \bar{k})$ for $t \varepsilon N$. Finally, define $Q(e, y) = \{(x, y, c) : (x, y, c)$ is an efficient program, and

$$\liminf_{T\to\infty} T^{-1} \sum_{0}^{T} u(c_t) \geq \liminf_{T\to\infty} T^{-1} \sum_{0}^{T} u(c'_t)$$

for all programs (x', y', c'), given $(e, y)\}$. Thus, the social goal is to attain an efficient program which also maximizes long-run average utility among all programs, given the environment, e, and the initial stock, y.

Bala, Majumdar and Mitra (1990) construct a decentralized evolutionary mechanism, which realizes the above social goal. The mechanism is based on a particularly simple version of continual planning revision, studied by Goldman and others. This mechanism $\pi = (M, \psi, H)$ is defined as follows. The message space at date t, $M_t = \mathbb{R}^6_{++}$ for $t \varepsilon N$; we write the generic element of M_t as $m_t = (x_t, c_t, x_{t+1}, d_{t+1}, y_{t+1}, r_t)$. Next, the equilibrium correspondence of the consumer at date t, ψ_{t_c}, is given by $\psi_{t_c}(e_{t_c}, y_t) = \{m_t \varepsilon M_t : u'(c_t) = r_t u'(d_{t+1})\}$; the equilibrium correspondence of the producer at date t, ψ_{t_R}, is given by $\psi_{t_R}(e_{t_R}, y_t) = \{m_t \varepsilon M_t : x_t + c_t = y_t, f(x_t) = y_{t+1}, x_{t+1} + d_{t+1} = y_{t+1}, f(x_{t+1}) = y_t, f'(x_t) = r_t\}$ for $t \varepsilon N$. And, $\psi_t(e_{t_c}, e_{t_R}, y_t) = \psi_{t_c}(e_{t_c}, y_t) \cap \psi_{t_R}(e_{t_R}, y_t)$ for $t \varepsilon N$. Finally, the outcome function, H_t, is given by $H_t(m_t) = (y_{t+1}, x_t, c_t)$ for $t \varepsilon N$.

It can be checked formally that π satisfies the definition of a decentralized evolutionary mechanism and π realizes the social goal defined above. [The details are supplied in Bala, Majumdar and Mitra (1990)]. We make the following somewhat informal remarks about the mechanism. In period t, given the message m_t, the consumer is asked to verify $[u'(c_t)/u'(d_{t+1})] = r_t$. The consumer, knowing u, can surely perform this verification, which is simply the equality of the marginal rate of substitution with an appropriate »shadow» price ratio, r_t. The producer is asked to verify $x_t + c_t = y_t$, $f(x_t) = y_{t+1}$, $x_{t+1} + d_{t+1} = y_{t+1}$, $f(x_{t+1}) = y_t$ and $f'(x_t) = r_t$. The first four conditions are verifications of feasibility. The final condition is the equality of the marginal rate of transformation with an appropriate »shadow» price ratio, r_t. The producer, knowing f and y_t, can perform these verifications. The above verifications imply that $x_t + c_t = y_t$, $f(x_t) = y_{t+1}$, $x_{t+1} + d_{t+1}$, $f(x_{t+1}) = y_t$ and $[u'(c_t)/u'(d_{t+1})] = f'(x_t)$. But, these conditions completely characterize a two period optimal plan for which the initial and terminal stock is y_t. Thus the state sequence generated by the mechanism is precisely the state sequence generated by continual planning revision of two-period optimal plans with initial and terminal stocks set equal to each other. Since a program generated by such a planning revision procedure is efficient and maximizes long-run average utility, so does the state-allocation sequence generated by this mechanism.

9. *Bibliographical Notes*

The sketch of mechanism theory in Section II is based on Hurwicz (1986), an excellent survey of a number of aspects of the literature on mechanism-design. Less technical and less formal reviews by Hurwicz (1973) and Reiter (1986) are also insightful. The results in Section III are well-known to specialists, but somewhat scattered. Many of the results have been proved in the multisector models. On the necessity of the transversality condition (IF) see Majumdar – Mitra – McFadden (1986) and Mitra – Majumdar (1976). For references to the literature on programs that are optimal in the sense of (3.14) see the reviews by Cass – Majumdar (1979) and McKenzie (1986). The assumption that u is continuous on $\mathbb{R}_+$ can be dispensed with for most of the results, and felicity functions like $u(c) = \log c$ can be handled. Specifically, Majumdar

(1988) or Hurwicz–Majumdar (1988) allow for such felicity functions. For multisector extensions of Proposition 4.2 in the discounted and undiscounted cases see Dasgupta and Mitra (1988a) and Brock and Majumdar (1988).

Hurwicz and Majumdar (1988) also contain some results where the production function f is linear, i.e., $f(x) = \rho x$ for $\rho > 0$. Dasgupta and Mitra (1988b) consider a class of multisector, linear models and explore the possibility of characterizing optimal programs in terms of »period-by-period» conditions. The results reported in Section V are new. Section VII is adapted from Hurwicz and Majumdar (1988). Section VIII is adapted from Bala, Majumdar and Mitra (1990).

Our discussion of intertemporal decentralization has been confined to the non-stochastic case. For a non-technical review of some of the problems that can arise in the stochastic framework, see Radner (1970). For a discussion of value and policy functions under uncertainty, see Majumdar, Mitra and Nyarko (1989). For results on price characterizations of optimal programs with a stochastic technology (analogous to our Propositions 4.1 and 4.2), see Zilcha (1976, 1978). A version of our Proposition 4.3 in the stochastic case (when future utilities are not discounted) is contained in Nyarko (1988).

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